

Classification of linearly compact simple $N=6$ 3-algebras

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Abstract

$N \leq 8$ 3-algebras have recently appeared in N -supersymmetric 3-dimensional Chern-Simons gauge theories. In our previous paper we classified linearly compact simple $N = 8$ n -algebras for any $n \geq 3$. In the present paper we classify linearly compact simple $N = 6$ 3-algebras, using their correspondence with simple linearly compact Lie superalgebras with a consistent short \mathbb{Z} -grading, endowed with a graded conjugation. We also briefly discuss $N = 5$ 3-algebras.

0 Introduction

In recent papers on N -supersymmetric 3-dimensional Chern-Simons gauge theories various types of 3-algebras have naturally appeared (see [12], [3], [4], [1], [2], ...).

Recall that a 3-algebra (also called a triple system [13]) is a vector space V with a ternary (or 3-)bracket $V^{\otimes 3} \rightarrow V$, $a \otimes b \otimes c \mapsto [a, b, c]$.

The 3-algebras that appear in supersymmetric 3-dimensional Chern-Simons theories satisfy certain symmetry conditions and a Jacobi-like identity, usually called the fundamental identity (very much like the Lie algebra bracket).

The simplest among them are 3-Lie algebras, for which the symmetry condition is the total anti-commutativity:

$$(0.1) \quad [a, b, c] = -[b, a, c] = -[a, c, b],$$

and the fundamental identity is:

$$(0.2) \quad [a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]].$$

(Of course, identity (0.2) simply says that for each $a, b \in V$, the endomorphism $D_{a,b}(x) = [a, b, x]$ is a derivation of the 3-algebra V , very much like the Jacobi identity for Lie algebras; in fact, this identity appears already in [13].)

The notion of a 3-Lie algebra generalizes to that of an n -Lie algebra for an arbitrary integer $n \geq 2$ in the obvious way. In this form they were introduced by Filippov in 1985 [10]. It was subsequently proved in Ling's thesis [17] that for each $n \geq 3$ there is only one simple finite-dimensional n -Lie algebra over an algebraically closed field of characteristic 0, by an analysis of the linear Lie algebra spanned by the derivations $D_{a,b}$ (for $n = 3$ this fact was independently proved in [10]). This unique simple n -Lie algebra is the vector product n -algebra O^n in an $n + 1$ -dimensional vector space [10], [17]. Recall that, endowing an $n + 1$ -dimensional vector space V with a non-degenerate symmetric

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bilinear form (\cdot, \cdot) and choosing a basis $\{a_i\}$ and the dual basis $\{a^i\}$, so that $(a_i, a^j) = \delta_{ij}$, the vector product of n vectors from the basis $\{a_i\}$ is defined by:

$$(0.3) \quad [a_{i_1}, \dots, a_{i_n}] = \epsilon_{i_1, \dots, i_{n+1}} a^{i_{n+1}},$$

where $\epsilon_{i_1, \dots, i_{n+1}}$ is a non-zero totally antisymmetric tensor, and extended by n -linearity.

In [9] we extended this classification to linearly compact n -Lie algebras. Recall that a linearly compact n -algebra is a topological n -algebra (i.e., the n -product is continuous) whose underlying topological vector space is linearly compact. The basic examples of linearly compact spaces over a field \mathbb{F} are the spaces of formal power series $\mathbb{F}[[x_1, \dots, x_n]]$, endowed with formal topology, or a direct sum of a finite number of such spaces (these include finite-dimensional vector spaces with discrete topology). Our result is that a complete list of simple linearly compact n -Lie algebras over an algebraically closed field \mathbb{F} of characteristic 0 for $n > 2$ consists of four n -algebras: the $n + 1$ -dimensional O^n and three infinite-dimensional, which we denoted by S^n , W^n , SW^n (see [9] for their construction).

Our method consists in associating to an n -Lie algebra \mathfrak{g} a Lie superalgebra $L = \bigoplus_{j=-1}^{n-1} L_j$ with a consistent \mathbb{Z} -grading, such that $L_{-1} = \Pi\mathfrak{g}$ (i.e., the vector space \mathfrak{g} with odd parity), satisfying the following properties:

$$(0.4) \quad [a, L_{-1}] = 0, a \in L_j, j \geq 0, \quad \text{imply} \quad a = 0 \quad (\text{transitivity});$$

$$(0.5) \quad \dim L_{n-1} = 1 \quad \text{unless all } n\text{-brackets are } 0;$$

$$(0.6) \quad [L_j, L_{-1}] = L_{j-1} \quad \text{for all } j;$$

$$(0.7) \quad [L_j, L_{n-j-1}] = 0 \quad \text{for all } j.$$

Provided that $n > 2$, the Lie superalgebra $[L, L] = \bigoplus_{j=-1}^{n-2} L_j$ is simple, hence the classification of simple linearly compact n -Lie algebras is thereby reduced to the known classification of simple linearly compact Lie superalgebras [14], [16] and their consistent \mathbb{Z} -gradings [15], [5].

Note that, given a consistently \mathbb{Z} -graded Lie superalgebra $L = \bigoplus_{j=-1}^{n-1} L_j$, satisfying (0.4)–(0.7), we can recover the n -bracket on $\mathfrak{g} = \Pi L_{-1}$ by choosing a non-zero $\mu \in L_{n-1}$ and letting:

$$[a_1, \dots, a_n] = [\dots [\mu, a_1], \dots, a_n].$$

The 3-Lie algebras appear in $N = 8$ supersymmetric 3-dimensional Chern-Simons theories, hence it is natural to call them the $N = 8$ 3-algebras. The next in the hierarchy of 3-algebras are those which appear in $N = 6$ supersymmetric 3-dimensional Chern-Simons theories (case $N = 7$ reduces to $N = 8$), which we shall call $N = 6$ 3-algebras. They are defined by the following axioms:

$$\begin{aligned} [a, b, c] &= -[c, b, a]; \\ [a, b, [x, y, z]] &= [[a, b, x], y, z] - [x, [b, a, y], z] + [x, y, [a, b, z]]. \end{aligned}$$

Note that any $N = 8$ 3-algebra is also an $N = 6$ 3-algebra. (It is unclear how to define $N = 6$ n -algebras for $n > 3$.)

The main goal of the present paper is the classification of simple linearly compact $N = 6$ 3-algebras over \mathbb{C} . The method again consists of associating to an $N = 6$ 3-algebra a \mathbb{Z} -graded Lie superalgebra, but in a different way (our construction in [9] uses total anti-commutativity in an essential way).

Given an $N = 6$ 3-algebra \mathfrak{g} , following Palmkvist [P], we associate to \mathfrak{g} a pair $(L = L_{-1} \oplus L_0 \oplus L_1, \varphi)$, where L is a consistently \mathbb{Z} -graded Lie superalgebra with $L_{-1} = \Pi\mathfrak{g}$ and φ is its automorphism, such that the following properties hold:

$$(0.8) \quad \text{transitivity} \quad (0.4);$$

$$(0.9) \quad [L_{-1}, L_1] = L_0;$$

$$(0.10) \quad \varphi(L_j) = L_{-j} \quad \text{and} \quad \varphi^2(a) = (-1)^j a \quad \text{if} \quad a \in L_j.$$

An automorphism φ of the \mathbb{Z} -graded Lie superalgebra L , satisfying (0.10), is called a *graded conjugation*.

It is easy to see that \mathfrak{g} is a simple $N = 6$ 3-algebra if and only if the associated Lie superalgebra L is simple. Thus, this construction reduces the classification of simple linearly compact $N = 6$ 3-algebras to the classification of consistent \mathbb{Z} -gradings of simple linearly compact Lie superalgebras of the form $L = L_{-1} \oplus L_0 \oplus L_1$ (then (0.8) and (0.9) automatically hold), and their graded conjugations φ . The 3-bracket on $\mathfrak{g} = \Pi L_{-1}$ is recovered by letting $[a, b, c] = [[a, \varphi(b)], c]$.

The resulting classification of $N = 6$ 3-algebras is given by Theorem 1.11, and its proof is given in Sections 2,3, and 4. In Section 5, based on the discussion in [2], we propose a definition and obtain a classification of $N = 5$ 3-algebras, and give a construction of a reduction from $N = 6$ 3-algebras to $N = 5$ 3-algebras.

The base field is an algebraically closed field \mathbb{F} of characteristic zero.

1 Examples of $N = 6$ 3-algebras

Definition 1.1 *An $N = 6$ 3-algebra is a 3-algebra whose 3-bracket $[\cdot, \cdot, \cdot]$ satisfies the following axioms:*

$$(a) \quad [u, v, w] = -[w, v, u]$$

$$(b) \quad [u, v, [x, y, z]] = [[u, v, x], y, z] - [x, [v, u, y], z] + [x, y, [u, v, z]]$$

Example 1.2 Every 3-Lie algebra is an $N = 6$ 3-algebra.

Example 1.3 Let A be an associative algebra and let $*$ be an anti-involution of A , i.e., for every $a \in A$, $(a^*)^* = a$ and for every $a, b \in A$, $(ab)^* = b^*a^*$. Then A with 3-bracket

$$(1.1) \quad [a, b, c] = ab^*c - cb^*a$$

is an $N = 6$ 3-algebra. For instance, if $A = M_n(\mathbb{F})$ and $*$ is the transposition map $^t : a \mapsto a^t$, then the corresponding 3-bracket (1.1) defines on A an $N = 6$ 3-algebra structure. Likewise, if $n = 2k$ and $*$ is the symplectic involution

$$* : a \mapsto J_{2k} a^t J_{2k}^{-1},$$

with $J_{2k} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, then (1.1) defines on $M_n(\mathbb{F})$ an $N = 6$ 3-algebra structure. If $n = 2$, this is in fact a 3-Lie algebra structure.

More generally, consider the vector space $M_{m,n}(\mathbb{F})$ of $m \times n$ matrices with entries in \mathbb{F} , and let $*$: $M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$ be a map satisfying the following property: for $v, u, y \in M_{m,n}(\mathbb{F})$,

$$(1.2) \quad (vu^*y)^* = y^*uv^*.$$

Then $M_{m,n}(\mathbb{F})$ with the 3-bracket (1.1) is an $N = 6$ 3-algebra. (Note that if $m = n$, then property (1.2) implies that $*$ is an anti-involution). As an example we can take $*$ = t ; we denote this 3-algebra by $A^3(m, n; t)$. Likewise, if $m = 2h$ and $n = 2k$ are even, it is easy to check that the map $st : M_{m,n}(\mathbb{F}) \rightarrow M_{m,n}(\mathbb{F})$, given by

$$(1.3) \quad st : a \mapsto J_{2k} a^t J_{2h}^{-1},$$

satisfies property (1.2), hence $M_{m,n}(\mathbb{F})$ with the corresponding 3-bracket (1.1) is an $N = 6$ 3-algebra, which we denote by $A^3(m, n; st)$. It is easy to see that the 3-algebra $A^3(2, 2; st)$ is actually an $N=8$ 3-algebra, isomorphic to O^3 .

Lemma 1.4 (a) Let $*$: $M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$ be defined by: $b^* = k^{-1} b^t h$ for some symmetric matrices $h \in GL_m(\mathbb{F})$ and $k \in GL_n(\mathbb{F})$, then the corresponding 3-bracket (1.1) is isomorphic to the one associated to transposition.

(b) Let $+$: $M_{2h,2k}(\mathbb{F}) \rightarrow M_{2k,2h}(\mathbb{F})$ be defined by: $a^+ = H_{2k} a^t H_{2h}^{-1}$ for some skew-symmetric matrices $H_{2i} \in GL_{2i}(\mathbb{F})$, then the corresponding 3-bracket (1.1) is isomorphic to the one associated to map (1.3).

Proof. (a) First of all notice that every symmetric matrix $A \in GL_n(\mathbb{F})$ can be written as the product $B^t B$ where $B \in GL_n(\mathbb{F})$ and B^t is its transpose. Hence let $h = x^t x \in GL_m(\mathbb{F})$ and $k = y^t y \in GL_n(\mathbb{F})$ and consider the map $\varphi : M_{m,n}(\mathbb{F}) \rightarrow M_{m,n}(\mathbb{F})$ defined by $\varphi(u) = x^{-1} u y$. For $a, b, c \in M_{m,n}(\mathbb{F})$, let $[a, b, c] = ab^t c - cb^t a$ and $[a, b, c]^* = ab^* c - cb^* a$. Then we have: $\varphi([a, b, c]) = \varphi(ab^t c - cb^t a) = x^{-1} ab^t c y - x^{-1} cb^t a y = (x^{-1} a y)(y^{-1} b^t x)(x^{-1} c y) - (x^{-1} c y)(y^{-1} b^t x)(x^{-1} a y) = \varphi(a) y^{-1} (y^{-1})^t (\varphi(b))^t x^t x \varphi(c) - \varphi(c) y^{-1} (y^{-1})^t (\varphi(b))^t x^t x \varphi(a) = \varphi(a) k^{-1} (\varphi(b))^t h \varphi(c) - \varphi(c) k^{-1} (\varphi(b))^t h \varphi(a) = [\varphi(a), \varphi(b), \varphi(c)]^*$, and this shows that the 3-brackets $[, ,]$ and $[, ,]^*$ are isomorphic.

(b) Now let $H_{2k} = B^{-1} J_{2k} B$ and $H_{2h} = A^{-1} J_{2h} A$ for some matrices $A \in GL_{2k}$ and $B \in GL_{2h}$ such that $B^{-1} = B^t$, $A^{-1} = A^t$. For $a, b, c \in M_{2h,2k}(\mathbb{F})$, let $[a, b, c]^* = ab^* c - cb^* a$ with $b^* = J_{2k} b^t J_{2h}^{-1}$. Let $\varphi : M_{2h,2k}(\mathbb{F}) \rightarrow M_{2h,2k}(\mathbb{F})$ be defined by $\varphi(x) = A^{-1} x B$. Then we have: $\varphi([a, b, c]^*) = \varphi(ab^* c - cb^* a) = A^{-1} ab^* c B - A^{-1} cb^* a B = (A^{-1} a B)(B^{-1} b^* A)(A^{-1} c B) - (A^{-1} c B)(B^{-1} b^* A)(A^{-1} a B) = \varphi(a) B^{-1} J_{2k} b^t J_{2h}^{-1} A \varphi(c) - \varphi(c) B^{-1} J_{2k} b^t J_{2h}^{-1} A \varphi(a) = \varphi(a) H_{2k} B^t b^t A H_{2h}^{-1} \varphi(c) - \varphi(c) H_{2k} B^t b^t A H_{2h}^{-1} \varphi(a) = \varphi(a) (H_{2k} \varphi(b)^t H_{2h}^{-1}) \varphi(c) - \varphi(c) (H_{2k} \varphi(b)^t H_{2h}^{-1}) \varphi(a) = [\varphi(a), \varphi(b), \varphi(c)]^+$, and this shows that the 3-brackets $[, ,]^*$ and $[, ,]^+$ are isomorphic. \square

Example 1.5 Let us consider the map $\psi : M_{1,2n}(\mathbb{F}) \rightarrow M_{2n,1}(\mathbb{F})$, defined by: $\psi(X Y) = (Y - X)^t$, for $X, Y \in M_{1,2n}$. Then $M_{1,2n}$ with 3-bracket

$$(1.4) \quad [A, B, C] = -AB^t C + CB^t A - C\psi(A)\psi(B)^t$$

is an $N = 6$ 3-algebra, which we denote by $C^3(2n)$.

The $N=6$ 3-algebras $A^3(m, n; t)$ and $C^3(2n)$ were introduced in [4].

Lemma 1.6 Let $[\cdot, \cdot, \cdot]_*$ be the 3-bracket on $M_{1,2n}$ defined by $[a, b, c]_* = -a(kb^t h^{-1})c + c(kb^t h^{-1})a - c\psi(a)h(\psi(b))^t k^{-1}$ for some matrix $h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in GL_2(\mathbb{F})$ and some symmetric matrix $k \in Sp_{2n}(\mathbb{F})$. Then $[\cdot, \cdot, \cdot]_*$ is isomorphic to (1.4).

Proof. Note that we can write $h = xx^t$ for some matrix $x \in GL_2(\mathbb{F})$ and $k = yy^t$ for some matrix $y \in Sp_{2n}(\mathbb{F})$.¹ Consider the map $\varphi : M_{1,2n} \rightarrow M_{1,2n}$ defined by: $\varphi(u) = xuy^{-1}$ and let $[\cdot, \cdot, \cdot]$ be 3-bracket (1.4). Then we have: $\varphi([a, b, c]) = -(xay^{-1})yb^tx^{-1}(xcy^{-1}) + (xcy^{-1})yb^tx^{-1}(xay^{-1}) - (xcy^{-1})y\psi(a)\psi(b)^ty^{-1} = -\varphi(a)yb^tx^{-1}\varphi(c) + \varphi(c)yb^tx^{-1}\varphi(a) - \varphi(c)y\psi(a)(\psi(b))^ty^{-1} = -\varphi(a)k\varphi(b)^th^{-1}\varphi(c) + \varphi(c)k\varphi(b)^th^{-1}\varphi(a) - \varphi(c)\psi(\varphi(a))h\psi(\varphi(b))^tk^{-1} = [\varphi(a), \varphi(b), \varphi(c)]_*$. In order to show that $y\psi(a)(\psi(b))^ty^{-1} = \psi(\varphi(a))h\psi(\varphi(b))^tk^{-1}$, it is convenient to identify $M_{1,2n}(\mathbb{F})$ with the set of matrices of the form $\begin{pmatrix} 0 & A \\ \psi(A) & 0 \end{pmatrix}$ where $A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$, with $a, b \in M_{1,n}$. Then, for $X \in M_{2,2n}(\mathbb{F})$, $\psi(X) = J_{2n}X^tJ_{2n}^{-1}$. Then we have: $\psi(\varphi(a))h\psi(\varphi(b))^tk^{-1} = J_{2n}(xay^{-1})^tJ_{2n}^{-1}xx^t(J_{2n}(xb y^{-1})^tJ_{2n}^{-1})^t(y^t)^{-1}y^{-1} = J_{2n}(y^{-1})^ta^tx^tJ_{2n}^{-1}xx^tJ_{2n}xy^{-1}J_{2n}^{-1}(y^t)^{-1}y^{-1} = yJ_{2n}a^tbJ_{2n}^{-1}y^{-1} = y\psi(a)\psi(b)^ty^{-1}$ since $y^tJ_{2n}y = J_{2n}$ and $x^tJ_2^{-1}hJ_2x = I_2$. \square

Example 1.7 Let P be a generalised Poisson algebra with bracket $\{\cdot, \cdot\}$ and derivation D (see [7] for the definition). Let σ be a Lie algebra automorphism of P such that $-\sigma$ is an associative algebra automorphism, $\sigma^2 = 1$ and $\sigma \circ D = -D \circ \sigma$. Then P with the 3-bracket:

$$(1.5) \quad [f, g, h] = \{f, \sigma(g)\}h + \{f, h\}\sigma(g) + f\{\sigma(g), h\} + D(f)\sigma(g)h - f\sigma(g)D(h),$$

is an $N = 6$ 3-algebra.

For example, consider the generalised Poisson algebra $P(m, 0)$ in the (even) indeterminates $p_1, \dots, p_k, q_1, \dots, q_k$ (resp. $p_1, \dots, p_k, q_1, \dots, q_k, t$) if $m = 2k$ (resp. $m = 2k + 1$) endowed with the bracket:

$$(1.6) \quad \{f, g\} = (2 - E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)(g) + \sum_{i=1}^k \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),$$

where $E = \sum_{i=1}^k (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$ (the first two terms in (1.6) vanish if m is even) and the derivation $D = 2 \frac{\partial}{\partial t}$ (which is 0 if m is even). Then the map $\sigma_\varphi : f(p_i, q_i) \mapsto -f(\varphi(p_i), \varphi(q_i))$ (resp. $\sigma_\varphi : f(t, p_i, q_i) \mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i))$), where φ is an involutive linear change of variables (i.e. $\varphi^2 = 1$), multiplying by -1 the 1-form $\sum_i (p_i dq_i - q_i dp_i)$ if m is even (resp. $dt + \sum_i (p_i dq_i - q_i dp_i)$ if m is odd), satisfies the conditions described above, hence the corresponding 3-bracket (1.5) defines on $P(m, 0)$ an $N = 6$ 3-algebra structure. We denote this 3-algebra by $P^3(m; \varphi)$.

Example 1.8 Let $A = \mathbb{F}[[x]]^{(1)} \oplus \mathbb{F}[[x]]^{(2)}$ be the direct sum of two copies of the algebra $\mathbb{F}[[x]]$; for $f \in \mathbb{F}[[x]]$, denote by $f^{(i)}$ the corresponding element in $\mathbb{F}[[x]]^{(i)}$. Set $D = d/dx$ and let $a = (a_{ij})$ be a matrix in $GL_2(\mathbb{F})$. We define the following 3-bracket:

$$[f^{(i)}, g^{(i)}, h^{(i)}] = (-1)^i a_{ij}((fD(h) - D(f)h)g(\varphi(x)))^{(i)}, \quad \text{with } j \neq i;$$

$$[f^{(i)}, g^{(j)}, h^{(i)}] = (-1)^i a_{jj}((fD(h) - D(f)h)g(\varphi(x)))^{(i)}, \quad \text{for } j \neq i;$$

$$[f^{(1)}, g^{(j)}, h^{(2)}] = a_{j1}((fD(g(\varphi(x))) - D(f)g(\varphi(x)))h)^{(1)} + a_{j2}(f(hD(g(\varphi(x))) - D(h)g(\varphi(x))))^{(2)}$$

and extend it to A by skew-symmetry in the first and third entries. If $a \in SL_2(\mathbb{F})$ and either $a^2 = -1$ and $\varphi = -1$, or $a^2 = 1$ and $\varphi = 1$, then $(A, [\cdot, \cdot, \cdot])$ is an $N = 6$ 3-algebra, which we denote by $SW^3(a)$. Note that if $a^2 = 1$, i.e., up to rescaling, $a = 1$, and $\varphi = 1$, then we get the $N = 8$ 3-algebra SW^3 [9].

¹As E. Vinberg explained to us, if σ is an anti-involution of a connected reductive group G and S denotes its fixed point set in G , then, by a well-known argument of Cartan, any element $k \in S$ can be represented in the form $k = y\sigma(y)$ for some $y \in G$, provided that S is connected.

Example 1.9 Let $A = \mathbb{F}[[x_1, x_2]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$. We consider the following 3-bracket:

$$(1.7) \quad [f, g, h] = \det \begin{pmatrix} f & \varphi(g) & h \\ D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \end{pmatrix}$$

where φ is an automorphism of the algebra A . If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with 3-bracket (1.7) is an $N = 6$ 3-algebra, which we denote by $S^3(\varphi)$. Note that if $\varphi = 1$, then we get the $N = 8$ 3-algebra S^3 [9]. Clearly, all 3-algebras $S^3(\varphi)$ with $\varphi \neq 1$ are isomorphic to each other.

Example 1.10 Let $A = \mathbb{F}[[x_1, x_2, x_3]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$. Consider the following 3-bracket on A :

$$(1.8) \quad [f, g, h] = \det \begin{pmatrix} D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \\ D_3(f) & D_3(\varphi(g)) & D_3(h) \end{pmatrix}$$

where φ is an automorphism of the algebra A . If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with product (1.8) is an $N = 6$ 3-algebra, which we denote by $W^3(\varphi)$. Note that if $\varphi = 1$, then we get the $N = 8$ 3-algebra W^3 [9]. Clearly, all 3-algebras $W^3(\varphi)$ with $\varphi \neq 1$ are isomorphic to each other.

One can check directly that the above examples are indeed $N = 6$ 3-algebras. However a proof of this without any computations will follow from the connection of $N = 6$ 3-algebras to Lie superalgebras, discussed below. The main result of the paper is the following theorem.

Theorem 1.11 *The following is a complete list of simple linearly compact $N = 6$ 3-algebras over \mathbb{C} :*

- (a) *finite-dimensional:* $A^3(m, n; t)$, $A^3(2m, 2n; st)$, $C^3(2n)$ ($m, n \geq 1$);
- (b) *infinite-dimensional:* $P^3(m; \varphi)$ ($m \geq 1$), $SW^3(a)$, $S^3(\varphi)$, $W^3(\varphi)$.

Proof. Theorem 2.3 from Section 2 reduces the classification in question to that of the pairs (L, σ) , where $L = L_{-1} \oplus L_0 \oplus L_1$ is a simple linearly compact Lie superalgebra with a consistent \mathbb{Z} -grading and σ is a graded conjugation of L . A complete list of possible such $L = L_{-1} \oplus L_0 \oplus L_1$ is given by Remarks 3.2 and 3.4 from Section 3. Finally, a complete list of graded conjugations of these L is given by Propositions 4.3 and 4.8 from Section 4.

By Theorem 2.3(b), the $N = 6$ 3-algebra is identified with ΠL_{-1} , on which the 3-bracket is given by the formula $[a, b, c] = [[a, \sigma(b)], c]$. This formula, applied to the \mathbb{Z} -graded Lie superalgebras L with graded conjugations, described by Proposition 4.3 (a), (b), (c) in the finite-dimensional case produces the 3-algebras $A^3(m, n; t)$, $A^3(2m, 2n; st)$, $C^3(2n)$, respectively, and those, described by Proposition 4.8(a) and (b), (c) and (d), (e) and (f), (g) in the infinite-dimensional case produces the 3-algebras $P^3(m; \varphi)$, $SW^3(a)$, $S^3(\varphi)$, $W^3(\varphi)$, respectively. The fact that all of them are indeed $N = 6$ 3-algebras follows automatically from Theorem 2.3(b). \square

2 Palmkvist's construction

Definition 2.1 *Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a Lie superalgebra with a consistent \mathbb{Z} -grading. A graded conjugation of \mathfrak{g} is a Lie superalgebra automorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that*

1. $\varphi(\mathfrak{g}_j) = \mathfrak{g}_{-j}$
2. $\varphi^2(x) = (-1)^k x$ for $x \in \mathfrak{g}_k$.

Theorem 2.2 Let $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$ be a \mathbb{Z} -graded consistent Lie superalgebra with a graded conjugation φ . Then the 3-bracket

$$[u, v, w] := [[u, \varphi(v)], w]$$

defines on $\Pi \mathfrak{g}_{-1}$ an $N = 6$ 3-algebra structure.

Proof. Since the grading of \mathfrak{g} is consistent, \mathfrak{g}_{-1} and \mathfrak{g}_1 are completely odd and \mathfrak{g}_0 is even. For $u, v, w \in \mathfrak{g}_{-1}$ we thus have: $[u, v, w] := [[u, \varphi(v)], w] = [u, [\varphi(v), w]] = -[[\varphi(v), w], u] = -[[w, \varphi(v)], u] = -[w, v, u]$, which proves property (a) in Definition 1.1.

Besides, for $u, v, x, y, z \in \mathfrak{g}_{-1}$ we have: $[u, v, [x, y, z]] - [x, y, [u, v, z]] = [[u, \varphi(v)], [[x, \varphi(y)], z]] - [[x, \varphi(y)], [[u, \varphi(v)], z]] = [[u, \varphi(v)], [x, \varphi(y)], z] + [[x, \varphi(y)], [[u, \varphi(v)], z]] - [[x, \varphi(y)], [[u, \varphi(v)], z]] = [[u, \varphi(v)], [x, \varphi(y)], z] = [[[u, \varphi(v)], x], \varphi(y)], z] + [[[x, [u, \varphi(v)], \varphi(y)], z] = [[[u, \varphi(v)], x], \varphi(y)], z] - [[[x, \varphi([u, \varphi(v)], y)], z] = [[[u, v, x], y, z] - [x, [v, u, y], z]$ \square

We shall now associate to an $N = 6$ 3-algebra T with 3-bracket $[\cdot, \cdot, \cdot]$, a \mathbb{Z} -graded Lie superalgebra $LieT = Lie_{-1}T \oplus Lie_0T \oplus Lie_1T$, as follows. For $x, y \in T$, denote by $L_{x,y}$ the endomorphism of T defined by $L_{x,y}(z) = [x, y, z]$. Besides, for $x \in T$, denote by φ_x the map in $Hom(\Pi T \otimes \Pi T, \Pi T)$ defined by $\varphi_x(y, z) = -[y, x, z]$. Here, as usual, ΠT denotes the vector space T with odd parity.

We let $Lie_{-1}T = \Pi T$, $Lie_0T = \langle L_{x,y} \mid x, y \in T \rangle$, $Lie_1T = \langle \varphi_x \mid x \in T \rangle$, and let $LieT = Lie_{-1}T \oplus Lie_0T \oplus Lie_1T$. Define the map $\sigma : LieT \rightarrow LieT$ by $(x, y, z \in T)$:

$$z \mapsto -\varphi_z, \quad \varphi_z \mapsto z, \quad L_{x,y} \mapsto -L_{y,x}.$$

Theorem 2.3 (a) $LieT$ is a \mathbb{Z} -graded Lie superalgebra with a short consistent grading, satisfying the following two properties:

(i) any non-zero \mathbb{Z} -graded ideal of $LieT$ has a non-zero intersection with both $Lie_{-1}T$ and Lie_1T ;

(ii) $[Lie_{-1}T, Lie_1T] = Lie_0T$.

(b) σ is a graded conjugation of the \mathbb{Z} -graded Lie superalgebra $LieT$ and the 3-product on T is recovered from the bracket on $LieT$ by the formula $[x, y, z] = [[x, \sigma(y)], z]$.

(c) The correspondence $T \rightarrow (LieT, \sigma)$ is bijective and functorial between $N = 6$ 3-algebras and the pairs $(LieT, \sigma)$, where $LieT$ is a \mathbb{Z} -graded Lie superalgebra with a short consistent grading, satisfying properties (i) and (ii), and σ is a graded conjugation of $LieT$.

(d) A 3-algebra T is simple (resp. finite-dimensional or linearly compact) if and only if $LieT$ is.

Proof. For $x, y, z \in T$, we have: $[\varphi_x, z] = -L_{z,x}$, and

$$(2.1) \quad [L_{x,y}, L_{x',y'}] = L_{[x,y,x'],y'} - L_{x',[y,x,y']}.$$

Note that $[L_{x,y}, \varphi_z] = -\varphi_{[y,x,z]}$. Finally, $[[\varphi_x, \varphi_y], z] = [\varphi_x, [\varphi_y, z]] + [\varphi_y, [\varphi_x, z]] = -[\varphi_x, L_{z,y}] - [\varphi_y, L_{z,x}] = [L_{z,y}, \varphi_x] + [L_{z,x}, \varphi_y] = -\varphi_{[y,z,x]} - \varphi_{[x,z,y]} = 0$. Hence $[\varphi_x, \varphi_y] = 0$.

It follows that $LieT$ is indeed a \mathbb{Z} -graded Lie superalgebra, satisfying (ii) and such that any non-zero ideal has a non-zero intersection with Lie_{-1} . It is straightforward to check (b), hence any non-zero ideal of $LieT$ has a non-zero intersection with Lie_1T , which completes the proof of (a).

(c) is clear by construction. Since the simplicity of T , by definition, means that all operators $L_{x,y}$ have no common non-trivial invariant subspace in T , it follows that T is a simple 3-algebra if and only if Lie_0T acts irreducibly on $Lie_{-1}T$. Hence, by the properties (i) and (ii) of $LieT$, T is simple if and only if $LieT$ is simple. The rest of (d) is clear as well. \square

Remark 2.4 $\sigma|_{Lie_0T} = 1$ if and only if T is a 3-Lie algebra.

3 Short gradings and graded conjugations

In this section we shall classify all short gradings of all simple linearly compact Lie superalgebras. We shall describe the short gradings of the classical Lie superalgebras \mathfrak{g} , $\mathfrak{g} \neq Q(n), P(n)$, in terms of linear functions f on the set of roots of \mathfrak{g} . For the description of the root systems of the classical Lie superalgebras, we shall refer to [15]. As for the Lie superalgebra $Q(n)$, we will denote by e_i and f_i the standard Chevalley generators of $Q(n)_{\bar{0}}$, and by \bar{e}_i and \bar{f}_i the corresponding elements in $Q(n)_{\bar{1}}$. Besides, we will identify $P(n)$ with the subalgebra of the Lie superalgebra $SHO(n, n)$ spanned by the following elements: $\{x_i x_j, \xi_i \xi_j : i, j = 1, \dots, n; x_i \xi_j : i \neq j = 1, \dots, n; x_i \xi_i - x_{i+1} \xi_{i+1} : i = 1, \dots, n-1\}$ (cf. [8, §8], and thus describe the \mathbb{Z} -gradings of $P(n)$ as induced by the \mathbb{Z} -gradings of $SHO(n, n)$. Recall that the Lie superalgebra $W(0, n)$ is simple for $n \geq 2$, and for $n = 2$ it is isomorphic to the classical Lie superalgebra $osp(2, 2)$ [14]. Moreover the Lie superalgebra $H(0, n)$ is simple for $n \geq 4$, and for $n = 4$ it is isomorphic to $psl(2, 2)$. Hence, when dealing with $W(0, n)$ (resp. $H(0, n)$) we shall always assume $n \geq 3$ (resp. $n \geq 5$). Likewise, since $W(1, 1) \cong K(1, 2)$ and $S(2, 1) \cong SKO(2, 3; 0)$ [5, §0], when dealing with $W(m, n)$ and $S(m, n)$ we shall always assume $(m, n) \neq (1, 1)$ and $(m, n) \neq (2, 1)$, respectively.

Proposition 3.1 *A complete list of simple finite-dimensional \mathbb{Z} -graded Lie superalgebras with a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same dimension, is, up to isomorphism, as follows:*

1. $A_m, B_m, C_m, D_m, E_6, E_7$ with the \mathbb{Z} -gradings, defined for each s such that $a_s = 1$ by $f(\alpha_s) = 1$, $f(\alpha_i) = 0$ for all $i \neq s$, where $\alpha_1, \dots, \alpha_m$ are simple roots and $\sum_i a_i \alpha_i$ is the highest root;
2. $psl(m, n)$ with the \mathbb{Z} -gradings defined by: $f(\epsilon_1) = \dots = f(\epsilon_k) = 1$, $f(\epsilon_{k+1}) = \dots = f(\epsilon_m) = 0$, $f(\delta_1) = \dots = f(\delta_h) = 1$, $f(\delta_{h+1}) = \dots = f(\delta_n) = 0$, for each $k = 1, \dots, m$ and $h = 1, \dots, n$;
3. $osp(2m+1, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta_j) = 0$ for all j ;
4. $osp(2, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\delta_j) = 0$ for all j ;
5. $osp(2, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1/2$, $f(\delta_j) = 1/2$ for all j ;
6. $osp(2m, 2n)$, $m \geq 2$, with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta_j) = 0$ for all j ;
7. $osp(2m, 2n)$, $m \geq 2$, with the \mathbb{Z} -grading defined by: $f(\epsilon_i) = 1/2$, $f(\delta_j) = 1/2$ for all i, j ;

8. $D(2, 1; \alpha)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = f(\epsilon_2) = 1/2$, $f(\epsilon_3) = 0$;
9. $F(4)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta) = 1$;
10. $Q(n)$ with the gradings defined by: $\deg(e_i) = \deg(\bar{e}_i) = -\deg(f_i) = -\deg(\bar{f}_i) = k_i$, with $k_s = 1$ for some s and $k_i = 0$ for all $i \neq s$;
11. $P(n)$ with $n = 2h \geq 2$ and the gradings of type $(1, \dots, 1, 0, \dots, 0 | 0, \dots, 0, 1, \dots, 1)$ with h 1's and h 0's both in the even and odd part;
12. $H(0, n)$ with the grading of type $(1, 0, \dots, 0, -1)$.

Proof. The claim for simple Lie algebras (the proof of which uses conjugacy of Borel subalgebras) is well known (see e.g. [15]).

In order to classify all short gradings of all classical Lie superalgebras \mathfrak{g} , we shall classify linear functions f on the set of roots of \mathfrak{g} which take values 1, 0 or -1 .

Let $\mathfrak{g} = psl(m, n)$, i.e. $= sl(m, n)$ for $m \neq n$, and $= sl(n, n)/FI_{n,n}$ for $m = n$. Since \mathfrak{g} has (even) roots $\pm(\epsilon_i - \epsilon_j)$ and $\pm(\delta_i - \delta_j)$, a linear function f on the set of roots taking values 0 and ± 1 , is defined, up to a permutation of ϵ_i 's and δ_j 's, by: $f(\epsilon_1) = \dots = f(\epsilon_k) = a$, $f(\epsilon_{k+1}) = \dots = f(\epsilon_m) = a - 1$, $f(\delta_1) = \dots = f(\delta_h) = b$, $f(\delta_{h+1}) = \dots = f(\delta_n) = b - 1$, for some $a, b, k = 0, \dots, m$, $h = 0, \dots, n$. On the other hand, \mathfrak{g} has (odd) roots $\pm(\epsilon_i - \delta_j)$, hence, either $b = a$ or $k = m$ and $b = a + 1$. Since the value of f on the roots is independent of a , we can let $a = 1$.

Now let $\mathfrak{g} = osp(2m + 1, 2n)$. Then \mathfrak{g} has roots δ_i and $2\delta_i$, hence $f(\delta_i) = 0$ for every i . It follows that for every j , either $f(\epsilon_j) = \pm 1$ or $f(\epsilon_j) = 0$. If $f(\epsilon_j) = 0$ for every j , then we get a grading which is not short, hence we can assume, up to equivalence, that $f(\epsilon_1) = 1$. Since \mathfrak{g} has roots $\epsilon_i \pm \epsilon_j$, it follows that $f(\epsilon_j) = 0$ for every $j \neq 1$.

Now let $\mathfrak{g} = osp(2, 2n)$. Then \mathfrak{g} has even roots of the form $\pm 2\delta_i$ and $\pm \delta_i \pm \delta_j$, and odd roots of the form $\pm \epsilon_1 \pm \delta_i$ hence, either $f(\delta_i) = 0$ or $f(\delta_i) = \pm 1/2$. If $f(\delta_i) = 0$ for every i , then $f(\epsilon_1) = 1$. If $f(\delta_k) = 1/2$ for some k , then $f(\delta_i) = 0$ for every i and $f(\epsilon_1) = 1/2$. We hence get two inequivalent short gradings.

Likewise, if $\mathfrak{g} = osp(2m, 2n)$ with $m \geq 2$, either $f(\delta_i) = 0$ for every i , $f(\epsilon_k) = 1$ for some k and $f(\epsilon_j) = 0$ for every $j \neq k$, or $f(\delta_i) = 0$ for every i and $f(\epsilon_j) = 1/2$ for every j .

Now let $\mathfrak{g} = D(2, 1; \alpha)$. Then \mathfrak{g} has roots $\pm 2\epsilon_i$, $i = 1, 2, 3$, hence we may assume that $f(\epsilon_1) = 1/2$. It follows that, up to equivalence, $f(\epsilon_2) = 1/2$ and $f(\epsilon_3) = 0$.

Let $\mathfrak{g} = F(4)$. Then \mathfrak{g} has even roots $\pm \epsilon_i \pm \epsilon_j$, $i \neq j$, $\pm \epsilon_i$, $\pm \delta$, and odd roots $1/2(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)$. It follows that there exists some k such that $f(\epsilon_k) = 1$, hence, $f(\epsilon_i) = 0$ for every $i \neq k$ and $f(\delta) = \pm 1$, i.e., up to equivalence, \mathfrak{g} has only one short grading.

Let $\mathfrak{g} = G(3)$. Since δ and 2δ are roots of \mathfrak{g} , we have $f(\delta) = 0$. Moreover, for every i , either $f(\epsilon_i) = 0$ or $f(\epsilon_i) = \pm 1$. If $f(\epsilon_i) = 0$ for every i , then the grading is not short, hence we may assume, up to equivalence, $f(\epsilon_1) = 1$. It follows that, for $j = 2, 3$, either $f(\epsilon_j) = 1$ or $f(\epsilon_j) = 0$. But this contradicts the linearity of f since $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Hence $G(3)$ has no short gradings.

Let $\mathfrak{g} = Q(n)$. Then any \mathbb{Z} -grading of \mathfrak{g} is defined by setting $\deg(e_i) = \deg(\bar{e}_i) = -\deg(f_i) = -\deg(\bar{f}_i) = k_i \in \mathbb{Z}_{\geq 0}$. Then, since $q(n)_{\bar{0}} \cong A_n$, it is clear that such a grading has depth one if and only if all k_i 's are 0 except for $k_s = 1$ for some s .

Finally, we recall that a complete list of \mathbb{Z} -gradings of depth 1 of all simple linearly compact Lie superalgebras is given in [8, Proposition 8.1]. Then if \mathfrak{g} is a simple finite-dimensional Lie superalgebra which is not classical, or $\mathfrak{g} = P(n)$, we select among these gradings the short ones

such that \mathfrak{g}_1 and \mathfrak{g}_{-1} have the same dimension, hence getting gradings 10. and 11. in the statement. \square

Remark 3.2 It follows from Proposition 3.1 that a complete list of simple finite-dimensional \mathbb{Z} -graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same dimension, is, up to isomorphism, as follows:

- $psl(m, n)$ with $m, n \geq 1$, $m + n \geq 2$, with the grading $f(\epsilon_1) = \dots = f(\epsilon_m) = 1$, $f(\delta_1) = \dots = f(\delta_n) = 0$;
- $osp(2, 2n)$, $n \geq 1$, with the grading $f(\delta_i) = 0$ for all i , $f(\epsilon_1) = 1$.

Notice that $P(n)$ has, up to equivalence, a unique consistent \mathbb{Z} -grading, i.e., the grading of type $(1, \dots, 1|0, \dots, 0)$. In this grading one has: $\mathfrak{g}_0 \cong sl_{n+1}$, $\mathfrak{g}_{-1} \cong \Lambda^2(\mathbb{F}^{n+1})^*$ and $\mathfrak{g}_1 \cong S^2\mathbb{F}^{n+1}$, where \mathbb{F}^{n+1} denotes the standard sl_{n+1} -module, hence \mathfrak{g}_1 and \mathfrak{g}_{-1} have different dimension. Finally, no short grading of the Lie superalgebra $Q(n)$ is consistent since in this case $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$ -module.

Proposition 3.3 *A complete list of simple linearly compact infinite-dimensional \mathbb{Z} -graded Lie superalgebras with a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth and the same size, is, up to isomorphism, as follows:*

- $S(1, 2)$ with the grading of type $(0|1, 0)$;
- $S(1, 2)$ with the grading of type $(0|1, 1)$;
- $H(2k, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- $K(2k + 1, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$;
- $E(1, 6)$ with the grading of type $(0|1, 0, 0, -1, 0, 0)$.

Proof. All \mathbb{Z} -gradings of depth 1 of all infinite-dimensional linearly compact simple Lie superalgebras are listed, up to isomorphism, in [8, Proposition 9.1]. Among these gradings we first select those which are short, hence getting the following list:

- 1) $W(m, n)$, with $m > 0$, $n \geq 1$, $(m, n) \neq (1, 1)$, with the grading of type $(0, \dots, 0|1, 0, \dots, 0)$;
- 2) $W(m, n)$, with $m > 0$, $n \geq 1$, $(m, n) \neq (1, 1)$, with the grading of type $(0, \dots, 0|-1, 0, \dots, 0)$;
- 3) $S(m, n)$, with $m > 1$ and $n \geq 1$, $(m, n) \neq (2, 1)$, or $m = 1$ and $n \geq 2$, with the grading of type $(0, \dots, 0|1, 0, \dots, 0)$;
- 4) $S(1, 2)$ with the grading of type $(0|1, 1)$;
- 5) $H(2k, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- 6) $K(2k + 1, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;

- 7) $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- 8) $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$;
- 9) $E(1, 6)$ with the grading of type $(0|1, 0, 0, -1, 0, 0)$.

Let us consider $W(m, n)$ with $n \geq 1$ and the grading of type $(0, \dots, 0|1, 0, \dots, 0)$. Then $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$ and $\mathfrak{g}_1 = \langle \xi_1 \frac{\partial}{\partial x_i}, \xi_1 \frac{\partial}{\partial \xi_j} | i = 1, \dots, m, j = 2, \dots, n \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$. Therefore \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth equal to m but \mathfrak{g}_{-1} has size 2^{n-1} and \mathfrak{g}_1 has size $(m + n - 1)2^{n-1}$. It follows that for $m > 0$, $n \geq 1$ and $(m, n) \neq (1, 1)$, \mathfrak{g}_{-1} and \mathfrak{g}_1 do not have the same size. Likewise case 2) is ruled out.

Now let us consider $S(m, n)$ with $n \geq 1$ and the grading of type $(0, \dots, 0|1, 0, \dots, 0)$. Then $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$ and $\mathfrak{g}_1 = \{f \in \langle \xi_1 \frac{\partial}{\partial x_i}, \xi_1 \frac{\partial}{\partial \xi_j} | i = 1, \dots, m, j = 2, \dots, n \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n) \mid \text{div}(f) = 0\}$. Therefore \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth equal to m but \mathfrak{g}_{-1} has size 2^{n-1} and \mathfrak{g}_1 has size $(m + n - 2)2^{n-1}$. It follows that for $m > 1$, $(m, n) \neq (2, 1)$, or $m = 1$ and $n \geq 2$, \mathfrak{g}_{-1} and \mathfrak{g}_1 do not have the same size unless $m = 1$ and $n = 2$. \square

Remark 3.4 It follows from Proposition 3.3 that a complete list of simple infinite-dimensional \mathbb{Z} -graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth and size, is, up to isomorphism, as follows:

- $S(1, 2)$ with the grading of type $(0|1, 1)$;
- $H(2k, 2)$ with the grading of type $(0, \dots, 0|1, -1)$;
- $K(2k + 1, 2)$ with the grading of type $(0, \dots, 0|1, -1)$;
- $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$.

4 Classification of graded conjugations

In this section we shall classify all graded conjugations σ of all \mathbb{Z} -graded simple linearly compact Lie superalgebras \mathfrak{g} with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In Lemma 4.7 and Proposition 4.8 we shall assume that $\mathbb{F} = \mathbb{C}$. This assumption can be removed with a little extra work.

Remark 4.1 If φ is an automorphism of \mathfrak{g} preserving the grading and σ is a graded conjugation, then $\varphi\sigma\varphi^{-1}$ is again a graded conjugation which is equivalent to σ . Indeed, we have: $\varphi\sigma\varphi^{-1}\varphi\sigma\varphi^{-1} = \varphi\mu\varphi^{-1} = \mu$, where $\mu|_{\mathfrak{g}_k} = (-1)^k \text{id}$.

Remark 4.2 If φ is an involution of \mathfrak{g} preserving the grading and commuting with σ , then $\varphi\sigma$ is again a graded conjugation.

Proposition 4.3 *The following is a complete list, up to equivalence, of graded conjugations of all simple finite-dimensional Lie superalgebras:*

$$(a) \ \mathfrak{g} = sl(m, n)/\mathbb{F}I\delta_{m, n}: \sigma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^t & c^t \\ -b^t & -d^t \end{pmatrix}.$$

(b) $\mathfrak{g} = sl(2h, 2k)/\mathbb{F}I\delta_{2h,2k}$: $\sigma_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^{st} & c^{st} \\ -b^{st} & -d^{st} \end{pmatrix}$, where st denotes the symplectic transposition defined by (1.3).

(c) $\mathfrak{g} = osp(2, 2n)$: σ_1 .

Proof. The Lie superalgebra $sl(m, n)$ has a short consistent grading such that $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, where $tr\alpha = tr\delta$, \mathfrak{g}_{-1} is the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$, and \mathfrak{g}_1 is the set of matrices of the form $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$.

For $m \neq n$ every automorphism of $sl(m, n)$ is either of the form $Ad \, diag(A, B)$ for some matrices $A \in GL_m(\mathbb{F})$, $B \in GL_n(\mathbb{F})$, or of the form $Ad \, diag(A, B) \circ \sigma_1$ [19]. Note that $\sigma_1^2 = Ad \, diag(I_m, -I_n)$ and $\sigma_1^3 = \sigma_1^{-1} = Ad \, diag(I_m, -I_n) \circ \sigma_1$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(m, n)$, we have: $Ad \, diag(A, B) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AaA^{-1} & AbB^{-1} \\ BcA^{-1} & BdB^{-1} \end{pmatrix}$, hence every automorphism $Ad \, diag(A, B)$ maps \mathfrak{g}_1 (resp. \mathfrak{g}_{-1}) to itself and does not define a graded conjugation of \mathfrak{g} . Let $\varphi_{A,B} = Ad \, diag(A, B) \circ \sigma_1$. Then $\varphi_{A,B} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -Aa^tA^{-1} & Ac^tB^{-1} \\ -Bb^tA^{-1} & -Bd^tB^{-1} \end{pmatrix}$. If $\varphi_{A,B}$ is a graded conjugation of \mathfrak{g} , then, by definition, $\varphi_{A,B}|_{\mathfrak{g}_{\bar{0}}}^2 = 1$, hence $A^tA^{-1} = \lambda I_m$ and $B^tB^{-1} = \rho I_n$, for some $\lambda, \rho \in \mathbb{F}$. Besides, since $\varphi_{A,B}|_{\mathfrak{g}_{\bar{1}}}^2 = -1$, we have $\rho = \lambda$. It follows that $A^t = \lambda A$ hence, by transposing both sides of the equality, $A = \lambda A^t = \lambda^2 A$, i.e., $\lambda^2 = 1$. Therefore, either $A^t = A$ and $B^t = B$, or $A^t = -A$, $B^t = -B$ and m and n are even (since A and B are invertible). The thesis then follows from Lemma 1.4.

If $m = n$, in addition to the automorphisms described above, $sl(n, n)/\mathbb{F}I_{n,n}$ has automorphisms of the form $Ad \, diag(A, B) \circ \Pi$, $Ad \, diag(A, B) \circ \Pi \circ \sigma_1$ and $Ad \, diag(A, B) \circ \sigma_1 \circ \Pi$, where $A, B \in GL_n(\mathbb{F})$, and Π is defined as follows: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(n, n)$, $\Pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ [19]. Note that $\sigma_1 \circ \Pi \circ \sigma_1 = \Pi$ and $\Pi \circ \sigma_1 \circ \Pi = \sigma_1^{-1}$. The automorphisms of the form $Ad \, diag(A, B) \circ \Pi \circ \sigma_1$ and $Ad \, diag(A, B) \circ \sigma_1 \circ \Pi$ map \mathfrak{g}_1 (resp. \mathfrak{g}_{-1}) to itself, hence they do not define graded conjugations of \mathfrak{g} . Let $\psi_{A,B} = Ad \, diag(A, B) \circ \Pi$. Then $\psi_{A,B} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AdA^{-1} & AcB^{-1} \\ BbA^{-1} & BaB^{-1} \end{pmatrix}$. It follows that $\psi_{A,B}|_{\mathfrak{g}_{\bar{0}}}^2 = 1$ if and only if $AB = BA = \lambda I_n$. As a consequence, $\psi_{A,B}|_{\mathfrak{g}_{\bar{1}}}^2 = 1$, hence $\psi_{A,B}$ does not define a graded conjugation.

The Lie superalgebra $osp(2, 2n)$ has a short consistent grading such that $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where $A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in \mathbb{F}$, and D lies in the Lie algebra $sp(2n)$,

defined by J_{2n} , \mathfrak{g}_{-1} is the set of matrices of the form $\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline b^t & 0 & a \\ -a^t & 0 & 0 \end{array} \right)$, and \mathfrak{g}_1 is the set of

matrices of the form $\left(\begin{array}{c|cc} 0 & a & b \\ \hline 0 & 0 & 0 \\ 0 & -a^t & 0 \end{array} \right)$, with $a, b \in M_{1,n}$.

Every automorphism of $osp(2, 2n)$ is either of the form $Ad \, diag(A, B)$ for some matrices $A = diag(\alpha, \alpha^{-1})$, $\alpha \in \mathbb{F}^\times$, $B \in Sp_{2n}(\mathbb{F})$, or of the form $Ad \, diag(A, B) \circ \sigma_1$ [19]. One can easily check that every automorphism of the form $Ad \, diag(A, B)$ sends \mathfrak{g}_{-1} (resp. \mathfrak{g}_1) to itself, hence it does not define a graded conjugation of \mathfrak{g} . Let $\Phi_{A,B} = Ad \, diag(A, B) \circ \sigma_1$. Then, using the same arguments as for the automorphisms $\varphi_{A,B}$ of the Lie superalgebra $sl(m, n)$, one can show that $\Phi_{A,B}$ defines a graded conjugation of $osp(2, 2n)$ if and only if B is a symmetric matrix. Then the result follows from Lemma 1.6. \square

Remark 4.4 If \mathfrak{g} is a simple infinite-dimensional linearly compact Lie superalgebra, then $Aut \, \mathfrak{g}$ contains a maximal reductive subgroup which is explicitly described in [6, Theorem 4.2]. We shall denote this subgroup by G . We point out that any reductive subgroup of $Aut \, \mathfrak{g}$ is conjugate into G , in particular any finite order element of $Aut \, \mathfrak{g}$ is conjugate to an element of G .

Example 4.5 The grading of type $(0, \dots, 0|1, -1)$ of $\mathfrak{g} = H(2k, 2)$ (resp. $K(2k+1, 2)$) is short. Let $A = \mathbb{F}[[p_1, \dots, p_k, q_1, \dots, q_k]]$ (resp. $A = \mathbb{F}[[t, p_1, \dots, p_k, q_1, \dots, q_k]]$). We have:

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle \xi_2 \rangle \otimes A, \\ \mathfrak{g}_0 &= (\langle 1, \xi_1 \xi_2 \rangle \otimes A) / \mathbb{F}1 \text{ (resp. } \langle 1, \xi_1 \xi_2 \rangle \otimes A), \\ \mathfrak{g}_1 &= \langle \xi_1 \rangle \otimes A. \end{aligned}$$

For every linear involutive change of variables φ , multiplying by -1 the 1-form $\sum_{i=1}^k (p_i dq_i - q_i dp_i)$ (resp. $dt + \sum_{i=1}^k (p_i dq_i - q_i dp_i)$), the following map is a graded conjugation of \mathfrak{g} :

$$(4.1) \quad \begin{aligned} f(p_i, q_i) &\mapsto -f(\varphi(p_i), \varphi(q_i)) & (\text{resp. } f(t, p_i, q_i) &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i))) \\ f(p_i, q_i) \xi_1 \xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i)) \xi_1 \xi_2 & (\text{resp. } f(t, p_i, q_i) \xi_1 \xi_2 &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_1 \xi_2) \\ f(p_i, q_i) \xi_1 &\mapsto f(\varphi(p_i), \varphi(q_i)) \xi_2 & (\text{resp. } f(t, p_i, q_i) \xi_1 &\mapsto f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_2) \\ f(p_i, q_i) \xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i)) \xi_1 & (\text{resp. } f(t, p_i, q_i) \xi_2 &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_1). \end{aligned}$$

Example 4.6 Let $\mathfrak{g} = S(1, 2)$, $SHO(3, 3)$, or $SKO(2, 3; 1)$. Then the algebra of outer derivations of \mathfrak{g} contains $sl_2 = \langle e, h, f \rangle$, with $e = \xi_1 \xi_2 \frac{\partial}{\partial x}$ and $h = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$ if $\mathfrak{g} = S(1, 2)$, $e = \xi_1 \xi_3 \frac{\partial}{\partial x_2} - \xi_2 \xi_3 \frac{\partial}{\partial x_1} - \xi_1 \xi_2 \frac{\partial}{\partial x_3}$ and $h = \sum_{i=1}^3 \xi_i \frac{\partial}{\partial \xi_i}$ if $\mathfrak{g} = SHO(3, 3)$, $e = \xi_1 \xi_2 \tau$ and $h = 1/2(\tau - x_1 \xi_1 - x_2 \xi_2)$ if $\mathfrak{g} = SKO(2, 3; 1)$. Let us denote by G_{out} the subgroup of $Aut \, \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(f))$ and $\exp(ad(h))$. We recall that $G_{out} \subset G$, where G is the subgroup of $Aut \, \mathfrak{g}$ introduced in Remark 4.4 [6, Remark 2.2, Theorem 4.2]. We shall denote by U_- the one parameter group of automorphisms $\exp(ad(tf))$, and by G_{inn} the subgroup of G consisting of inner automorphisms. Finally, H will denote the subgroup of $Aut \, \mathfrak{g}$ consisting of invertible changes of variables multiplying the volume form (resp. the even supersymplectic form) by a constant if $\mathfrak{g} = S(1, 2)$ (resp. $\mathfrak{g} = SHO(3, 3)$), or the odd supercontact form by a function if $\mathfrak{g} = SKO(2, 3; 1)$ (see [6, Theorem 4.5]).

The gradings of type $(0|1, 1)$, $(0, 0, 0|1, 1, 1)$ and $(0, 0|1, 1, 1)$ of $\mathfrak{g} = S(1, 2)$, $SHO(3, 3)$ and $SKO(2, 3; 1)$, respectively, are short, and the subspaces \mathfrak{g}_i 's are as follows:

$$\begin{aligned} \mathfrak{g} = S(1, 2): \\ \mathfrak{g}_{-1} &= \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes \mathbb{F}[[x]] \\ \mathfrak{g}_0 &= \{f \in \langle \frac{\partial}{\partial x}, \xi_i \frac{\partial}{\partial \xi_j} | i, j = 1, 2 \rangle \otimes \mathbb{F}[[x]], \text{div}(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \xi_i \frac{\partial}{\partial x}, \xi_1 \xi_2 \frac{\partial}{\partial \xi_i} | i = 1, 2 \rangle \otimes \mathbb{F}[[x]], \text{div}(f) = 0\}. \\ \mathfrak{g} = SHO(3, 3): \\ \mathfrak{g}_{-1} &= \mathbb{F}[[x_1, x_2, x_3]] / \mathbb{F}1 \end{aligned}$$

$$\begin{aligned}\mathfrak{g}_0 &= \{f \in \langle \xi_1, \xi_2, \xi_3 \rangle \otimes \mathbb{F}[[x_1, x_2, x_3]] | \Delta(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \xi_i \xi_j, i, j = 1, 2, 3 \rangle \otimes \mathbb{F}[[x_1, x_2, x_3]] | \Delta(f) = 0\}.\end{aligned}$$

$\mathfrak{g} = SKO(2, 3; 1)$:

$$\begin{aligned}\mathfrak{g}_{-1} &= \mathbb{F}[[x_1, x_2]] \\ \mathfrak{g}_0 &= \{f \in \langle \xi_1, \xi_2, \tau \rangle \otimes \mathbb{F}[[x_1, x_2]] | \text{div}_1(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \tau \xi_i, \xi_1 \xi_2 \mid i = 1, 2 \rangle \otimes \mathbb{F}[[x_1, x_2]] | \text{div}_1(f) = 0\}.\end{aligned}$$

In all these cases the map $s = \exp(\text{ad}(e)) \exp(\text{ad}(-f)) \exp(\text{ad}(e))$ is a graded conjugation of \mathfrak{g} : for $z \in \mathfrak{g}_{-1}$, $s(z) = [e, z]$; for $z \in \mathfrak{g}_1$, $s(z) = -[f, z]$, for $z \in \mathfrak{g}_0$, $s(z) = z$. Note that each of the above gradings can be extended to $\text{Der } \mathfrak{g} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{a} \supset \mathfrak{sl}_2$, so that e has degree 2, h has degree 0, and f has degree -2 .

Lemma 4.7 *If \mathfrak{g} is one of the following \mathbb{Z} -graded Lie superalgebras:*

1. $S(1, 2)$ with the grading of type $(0|1, 1)$,
2. $SKO(2, 3; 1)$ with the grading of type $(0, 0|1, 1, 1)$,
3. $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$,

and σ is a graded conjugation of \mathfrak{g} , then σ is conjugate to an automorphism of the form $s \circ \exp(\text{ad}(th)) \circ \varphi$, for some $t \in \mathbb{F}$ and some $\varphi \in G_{\text{inn}}$ such that $\varphi^2 = 1$.

Proof. Let us first assume $\mathfrak{g} = S(1, 2)$ with the grading of type $(0|1, 1)$, or $\mathfrak{g} = SKO(2, 3; 1)$ with the grading of type $(0, 0|1, 1, 1)$. By [6, Remark 4.6], if ψ is an automorphism of \mathfrak{g} lying in G , then either $\psi \in U_- H \cap G$ or $\psi \in U_- sH \cap G$. Note that $U_- H \cap G = U_-(H \cap G)$ and $U_- sH \cap G = U_- s(H \cap G)$, since $U_- \subset G$ and $s \in G$ [6, Theorem 4.2]. Here $H \cap G$ is the subgroup of $\text{Aut } \mathfrak{g}$ generated by $\exp(\text{ad}(e))$, $\exp(\text{ad}(h))$ and G_{inn} . Note that $G_{\text{inn}} \subset \exp(\text{ad}(\mathfrak{g}_0))$. Let $\psi \in U_-(H \cap G)$. Then $\psi = \exp(\text{ad}(tf))\psi_0$ for some $t \in \mathbb{F}$ and some $\psi_0 \in H \cap G$. For $x \in \mathfrak{g}_1$, we have: $\psi(x) = \exp(\text{ad}(tf))(\psi_0(x)) = \psi_0(x) + t[f, \psi_0(x)]$, since $\psi_0(x) \in \mathfrak{g}_1$. In particular, $\psi(x) \notin \mathfrak{g}_{-1}$. Now let σ be a graded conjugation of \mathfrak{g} . Then we may assume, up to conjugation, that σ lies in G . Since σ exchanges \mathfrak{g}_1 and \mathfrak{g}_{-1} , by the observation above $\sigma \in U_- s(H \cap G)$, i.e., $\sigma = \exp(\text{ad}(tf))s\varphi_0\varphi_1$ for some $t \in \mathbb{F}$, some $\varphi_1 \in G_{\text{inn}}$ and some φ_0 lying in the subgroup generated by $\exp(\text{ad}(h))$ and $\exp(\text{ad}(e))$. We can assume $\varphi_0 = \exp(\text{ad}(\beta e)) \exp(\text{ad}(\alpha h))$ for some $\alpha, \beta \in \mathbb{F}$, i.e., $\sigma = \exp(\text{ad}(tf))s \exp(\text{ad}(\beta e)) \exp(\text{ad}(\alpha h))\varphi_1$. Since $\varphi_1(\mathfrak{g}_1) = \mathfrak{g}_1$ and $\varphi_1(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$, for $x \in \mathfrak{g}_1$ we have:

$$\sigma(x) = -\exp(\alpha)[f, \varphi_1(x)] = s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1(x).$$

For $x \in \mathfrak{g}_{-1}$ we have:

$$(4.2) \quad \sigma(x) = \exp(-\alpha)([e, \varphi_1(x)] + (t - \beta)\varphi_1(x)).$$

Notice that if $x \in \mathfrak{g}_{-1}$, then $[e, \varphi_1(x)] \in \mathfrak{g}_1$ and $\varphi_1(x) \in \mathfrak{g}_{-1}$. Since $\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_1$, we have

$$(4.2) = \exp(-\alpha)[e, \varphi_1(x)] = s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1(x).$$

Therefore $\sigma = s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1$. Now notice that $s \circ \exp(\text{ad}(\alpha h)) = \exp(\text{ad}(-\alpha h)) \circ s$, $\exp(\text{ad}(\alpha h)) \circ \varphi_1 = \varphi_1 \circ \exp(\text{ad}(\alpha h))$, and $s \circ \varphi_1 = \varphi_1 \circ s$. It follows that $\sigma^2 = s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1 \circ s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1 = s \circ \exp(\text{ad}(\alpha h)) \exp(\text{ad}(-\alpha h)) \circ s \circ \varphi_1^2 = s^2 \circ \varphi_1^2$, therefore $\varphi_1^2 = 1$.

Now let $\mathfrak{g} = SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$. As in the previous cases, if σ is a graded conjugation of \mathfrak{g} , then $\sigma \in U_-s(H \cap G)$. Here $H \cap G$ is the subgroup of $Aut \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(h))$, $\exp(ad(\Phi))$, and G_{inn} , where $\Phi = \sum_{i=1}^3 (-x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ and G_{inn} is generated by $\exp(ad(x_i \xi_j))$ with $i, j = 1, 2, 3$, $i \neq j$, and is thus isomorphic to SL_3 . Note that Φ commutes with G_{inn} and h . We may hence assume $\sigma = \exp(ad(tf))s\varphi_0\varphi_1$, for some $t \in \mathbb{F}$, some $\varphi_1 \in G_{inn}$ and some φ_0 lying in the subgroup generated by $\exp(ad(e))$, $\exp(ad(h))$, $\exp(ad(\Phi))$, i.e., $\varphi_0 = \exp(ad(\gamma e)) \circ \exp(ad(\beta\Phi)) \circ \exp(ad(\alpha h))$, for some $\alpha, \beta, \gamma \in \mathbb{F}$. Arguing as for $S(1, 2)$ and $SKO(2, 3; 1)$, one shows that, in fact, $\sigma = s \circ \exp(ad(\beta\Phi)) \circ \exp(ad(\alpha h)) \circ \varphi_1$. Besides, the following commutation relations hold: $s \circ \exp(ad(\alpha h)) = \exp(ad(-\alpha h)) \circ s$, $\exp(ad(\alpha h)) \circ \varphi_1 = \varphi_1 \circ \exp(ad(\alpha h))$, $s \circ \varphi_1 = \varphi_1 \circ s$, $\exp(ad(\beta\Phi)) \circ s|_{\mathfrak{g}_{-1}} = \exp(3\beta)s \circ \exp(ad(\beta\Phi))|_{\mathfrak{g}_{-1}}$, $\exp(ad(\beta\Phi)) \circ s|_{\mathfrak{g}_1} = \exp(-3\beta)s \circ \exp(ad(\beta\Phi))|_{\mathfrak{g}_1}$. It follows that, since $\sigma^2 = s^2$, we have: $\exp(3\beta)\exp(ad(\beta\Phi))^2\varphi_1^2|_{\mathfrak{g}_{-1}} = 1$, $\exp(-3\beta)\exp(ad(\beta\Phi))^2\varphi_1^2|_{\mathfrak{g}_1} = 1$. Note that G_{inn} acts on \mathfrak{g}_{-1} by the standard action of vector fields on functions. In particular $V = \langle x_1, x_2, x_3 \rangle$ is stabilized by this action and $\exp(ad(\Phi))$ acts on V by scalar multiplication by $\exp(-1)$. It follows that if $F \in SL_3$ is the matrix of the action of φ_1 on V , then $\exp(\beta)I_3 F^2 = I_3$, hence $\exp(3\beta) = \exp(3\beta)\det(F)^2 = 1$. It follows that $(\exp(ad(\beta\Phi)) \circ \varphi_1)^2 = 1$, moreover, we can assume $\beta = 0$, since $\exp(\beta)I_3 \in SL_3$. Hence $\sigma = s \circ \exp(ad(\alpha h)) \circ \varphi_1$, for some $\alpha \in \mathbb{F}$ and some $\varphi_1 \in G_{inn}$ such that $\varphi_1^2 = 1$. \square

Proposition 4.8 *The following is a complete list, up to equivalence, of graded conjugations of all simple infinite-dimensional linearly compact Lie superalgebras \mathfrak{g} :*

- a) $\mathfrak{g} = H(2k, 2)$: σ is the automorphism of \mathfrak{g} defined by (4.1).
- b) $\mathfrak{g} = K(2k + 1, 2)$: σ is the automorphism of \mathfrak{g} defined by (4.1).
- c) $\mathfrak{g} = S(1, 2)$: s .
- d) $\mathfrak{g} = S(1, 2)$: $\sigma = s \circ \exp(ad(\alpha h')) \circ \varphi_0$, where $\exp(2\alpha) = -1$, $h' = 2x \frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$, φ_0 lies in the SL_2 -subgroup of G_{inn} , and $\varphi_0^2 = -1$.
- e) $\mathfrak{g} = SKO(2, 3; 1)$: s .
- f) $\mathfrak{g} = SKO(2, 3; 1)$: $\sigma = s \circ \exp(ad(\alpha h'))$, where $\exp(2\alpha) = -1$ and $h' = x_1 \xi_1 + x_2 \xi_2 + \tau$.
- g) $\mathfrak{g} = SHO(3, 3)$: $\sigma = s \circ \varphi$ with $\varphi \in G_{inn}$ such that $\varphi^2 = 1$.

Proof. By definition of graded conjugation, \mathfrak{g} is, up to isomorphism, one of the \mathbb{Z} -graded Lie superalgebras listed in Remark 3.4. Let $\mathfrak{g} = H(2k, 2)$ with the grading of type $(0, \dots, 0|1, -1)$ (see Example 4.5), and let σ be a graded conjugation of \mathfrak{g} . By Remark 4.4 we can assume that $\sigma \in G = \mathbb{F}^\times (Sp_{2k} \times O_2)$ [6, Theorem 4.2]. Note that G consists of linear changes of variables preserving the symplectic form up to multiplication by a non-zero scalar. Since σ exchanges \mathfrak{g}_{-1} and \mathfrak{g}_1 , and $\sigma^2|_{\mathfrak{g}_{-1}} = -1$, we have: $\sigma(\xi_1) = a\xi_2$ and $\sigma(\xi_2) = -\frac{1}{a}\xi_1$, for some $a \in \mathbb{F}^\times$, hence, up to multiplication by a scalar, we may assume that $a = 1$. It follows that, for $f \in \mathbb{F}[[p_i, q_i]]$, $\sigma(f) = \sigma([\xi_1, f\xi_2]) = [\xi_2, \sigma(f\xi_2)] = [\xi_2, \tilde{f}\xi_1] = \tilde{f}$ for some $\tilde{f} \in \mathbb{F}[[p_i, q_i]]$, i.e., $\sigma(f\xi_2) = \sigma(f)\xi_1$. Likewise, $\sigma(f\xi_1) = -\sigma(f)\xi_2$, and $\sigma(f\xi_1\xi_2) = \sigma(f)\xi_1\xi_2$. Besides, for $f, g \in \mathbb{F}[[p_i, q_i]]$ $\sigma([f\xi_1, g\xi_2]) = \sigma(fg + [f, g]\xi_1\xi_2) = -[\sigma(f)\xi_2, \sigma(g)\xi_1] = -\sigma(f)\sigma(g) + [\sigma(f), \sigma(g)]\xi_1\xi_2$. Hence $\sigma(fg) = -\sigma(f)\sigma(g)$,

i.e., $-\sigma$ is an automorphism of $\mathbb{F}[[p_i, q_i]]$ as an associative algebra. It follows that σ is defined as follows:

$$(4.3) \quad \begin{aligned} f(p_i, q_i) &\mapsto -f(\varphi(p_i), \varphi(q_i)) \\ f(p_i, q_i)\xi_1\xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i))\xi_1\xi_2 \\ f(p_i, q_i)\xi_1 &\mapsto f(\varphi(p_i), \varphi(q_i))\xi_2 \\ f(p_i, q_i)\xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i))\xi_1 \end{aligned}$$

for some linear change of even variables φ . Since $\sigma(\xi_1) = \xi_2$ and $\sigma(\xi_2) = -\xi_1$, σ multiplies the odd part $d\xi_1 d\xi_2$ of the symplectic form by -1 , hence φ multiplies the even part of the symplectic form by -1 . Moreover, $\sigma^2(f(p_i, q_i)) = f(\varphi^2(p_i), \varphi^2(q_i))$, hence $\varphi^2 = 1$. This concludes the proof of a). The same arguments prove b). In this case, one has $\sigma \circ \frac{\partial}{\partial t} = -\frac{\partial}{\partial t} \circ \sigma$.

Let $\mathfrak{g} = SKO(2, 3; \beta)$ with $\beta \neq 1$ and the grading of type $(0, 0|1, 1, 1)$. By [6, Theorem 4.2], G is generated by $\exp(ad(\tau + x_1\xi_1 + x_2\xi_2))$ and G_{inn} . Note that G_{inn} is contained in $\exp(ad(\mathfrak{g}_0))$, hence no automorphism of \mathfrak{g} exchanges \mathfrak{g}_1 and \mathfrak{g}_{-1} . It follows that \mathfrak{g} has no graded conjugations.

Let $\mathfrak{g} = S(1, 2)$ and let σ be a graded conjugation of \mathfrak{g} . Then, by Lemma 4.7, $\sigma = s \circ \exp(ad(th)) \circ \varphi$ for some $t \in \mathbb{F}$ and some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$. The group G_{inn} is generated by $\exp(ad(h'))$, $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_2}))$, $\exp(ad(\xi_2 \frac{\partial}{\partial \xi_1}))$ and $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}))$, hence we may write $\varphi = \exp(ad(\alpha h'))\varphi_0$ for some $\alpha \in \mathbb{F}$ and some φ_0 in the SL_2 -subgroup of G_{inn} generated by $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_2}))$, $\exp(ad(\xi_2 \frac{\partial}{\partial \xi_1}))$ and $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}))$. Note that $\varphi(\frac{\partial}{\partial x}) = \exp(-2\alpha)\frac{\partial}{\partial x}$, therefore $\exp(2\alpha) = \pm 1$ since $\varphi^2 = 1$. Besides, if $z \in \mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes \mathbb{F}[[x]]$, then $\varphi^2(z) = \exp(-2\alpha)\varphi_0^2(z)$, therefore either

$$i) \exp(2\alpha) = 1 \text{ and } \varphi_0^2|_{\mathfrak{g}_{-1}} = 1;$$

or

$$ii) \exp(2\alpha) = -1 \text{ and } \varphi_0^2|_{\mathfrak{g}_{-1}} = -1.$$

In case $i)$ we have $\varphi_0|_{\mathfrak{g}_{-1}} = \pm 1$, since $\varphi_0 \in SL_2$. Then $\varphi_0|_{\mathfrak{g}_0} = 1$ since φ_0 acts on \mathfrak{g}_0 via the adjoint action. It follows that $\sigma|_{\mathfrak{g}_0} = 1$, since $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h'))\varphi_0$, $\exp(ad(\alpha h'))|_{\mathfrak{g}_0} = 1$ since $\exp(2\alpha) = 1$, $\exp(ad(th))|_{\mathfrak{g}_0} = 1$ and $s|_{\mathfrak{g}_0} = 1$. By Remark 2.4 and the classification of 3-Lie algebras obtained in [9], we conclude that σ is conjugate to s .

In case $ii)$ φ_0 corresponds to a 2×2 matrix of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ such that $a^2 + bc = -1$.

The corresponding element $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h'))\varphi_0$ acts on \mathfrak{g}_{-1} as follows:

$$\begin{aligned} x^r \frac{\partial}{\partial \xi_1} &\mapsto \exp(2r\alpha) \exp(-t - \alpha) (-arx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_1} - brx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_2} + ax^r\xi_2 \frac{\partial}{\partial x} - bx^r\xi_1 \frac{\partial}{\partial x}); \\ x^r \frac{\partial}{\partial \xi_2} &\mapsto \exp(2r\alpha) \exp(-t - \alpha) (-crx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_1} + arx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_2} + cx^r\xi_2 \frac{\partial}{\partial x} + ax^r\xi_1 \frac{\partial}{\partial x}). \end{aligned}$$

It follows that, up to rescaling, we may assume that $t = 0$ hence getting d).

In order to classify the graded conjugations of $\mathfrak{g} = SKO(2, 3; 1)$ we argue in a similar way as for $S(1, 2)$. Namely, let σ be such a map, then, by Lemma 4.7, $\sigma = s \circ \exp(ad(th)) \circ \varphi$ for some $t \in \mathbb{F}$ and some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$. The group G_{inn} is generated by $\exp(ad(h'))$, $\exp(ad(x_1\xi_2))$, $\exp(ad(x_2\xi_1))$ and $\exp(ad(x_1\xi_1 - x_2\xi_2))$, hence we may write $\varphi = \exp(ad(\alpha h'))\varphi_0$ for some $\alpha \in \mathbb{F}$ and some φ_0 in the SL_2 -subgroup of G_{inn} generated by $\exp(ad(x_1\xi_2))$, $\exp(ad(x_2\xi_1))$ and $\exp(ad(x_1\xi_1 - x_2\xi_2))$. Note that this SL_2 -subgroup acts on $\mathfrak{g}_{-1} = \mathbb{F}[[x_1, x_2]]$ via the standard action of vector fields on functions, and stabilizes the subspaces of $\mathbb{F}[[x_1, x_2]]$ consisting of homogeneous polynomials of

fixed degree. We have: $\varphi(1) = \exp(\text{ad}(\alpha h'))(1) = \exp(-2\alpha)$, therefore, since $\varphi^2 = 1$, either $\exp(2\alpha) = 1$ or $\exp(2\alpha) = -1$. If $\exp(2\alpha) = 1$, then, for $f \in \mathfrak{g}_{-1}$, $\varphi(f) = \varphi_0(f)$; if $\exp(2\alpha) = -1$, then, for $f \in \mathfrak{g}_{-1}$, $\varphi(f) = -\varphi_0(f^-)$, where $f^-(x_1, x_2) = f(-x_1, -x_2)$. It follows that $\varphi_0^2|_{\mathfrak{g}_{-1}} = 1$. In particular, if $V = \langle x_1, x_2 \rangle$, then $\varphi_0|_V = \pm 1$, i.e., $\varphi_0 = \exp(\text{ad}(A(x_1\xi_1 - x_2\xi_2)))$ with $A \in \mathbb{F}$ such that $\exp(A) = \pm 1$. It follows that $\sigma = s \circ \exp(\text{ad}(th)) \circ \exp(\text{ad}(\alpha h')) \exp(\text{ad}(A(x_1\xi_1 - x_2\xi_2)))$, for some $t, \alpha, A \in \mathbb{F}$ such that $\exp(2\alpha) = \pm 1$, $\exp(A) = \pm 1$. We now consider the restriction of σ to \mathfrak{g}_0 . We have: $\sigma|_{\mathfrak{g}_0} = \exp(\text{ad}(\alpha h')) \exp(\text{ad}(A(x_1\xi_1 - x_2\xi_2)))|_{\mathfrak{g}_0}$, since $s|_{\mathfrak{g}_0} = 1$ and $\exp(\text{ad}(th))|_{\mathfrak{g}_0} = 1$. It is then easy to check that either

$$i) \exp(2\alpha) \exp(A) = 1 \text{ and } \sigma|_{\mathfrak{g}_0} = 1;$$

or

$$ii) \exp(2\alpha) \exp(A) = -1.$$

In case *i*), by Remark 2.4 and the classification of 3-Lie algebras obtained in [9], σ is conjugate to s . In case *ii*), $\sigma = s \circ \exp(\text{ad}(th)) \circ \exp(\text{ad}(\alpha h')) \exp(\text{ad}(A(x_1\xi_1 - x_2\xi_2)))$ acts on $\mathfrak{g}_{-1} = \mathbb{F}[[x_1, x_2]]$ as follows:

$$f \mapsto -\exp(-t)s(f^-), \quad \text{if } \exp(A) = 1, \exp(-2\alpha) = -1;$$

$$f \mapsto \exp(-t)s(f^-), \quad \text{if } \exp(A) = -1, \exp(-2\alpha) = 1.$$

Therefore, changing the sign if necessary, we may assume that we are in the first case and in this case we may assume, up to rescaling, that $A = 0 = t$, hence getting f).

Let $\mathfrak{g} = SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$. Then, by Lemma 4.7, $\sigma = s \circ \exp(\text{ad}(th)) \circ \varphi$ for some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$, and some $t \in \mathbb{F}$. For $a \in \mathfrak{g}_{-1}$, we have:

$$\sigma(a) = s(\exp(\text{ad}(th))(\varphi(a))) = \exp(-t)s(\varphi(a)) = \exp(-t)[e, \varphi(a)],$$

since $\varphi(a) \in \mathfrak{g}_{-1}$. Up to rescaling, we can thus assume that $t = 0$, hence getting g). Note that in this case $G_{inn} \cong SL_3$. \square

Remark 4.9 It was proved in [9] that there are no simple linearly compact $N = 8$ 3-superalgebras, which are not 3-algebras. On the contrary, there are many simple linearly compact $N = 6$ 3-superalgebras beyond 3-algebras. We are planning to classify them in a subsequent publication.

5 $N = 5$ 3-algebras

Based on the discussion in [2], the following seems to be a right definition of an $N = 5$ 3-algebra.

Definition 5.1 *An $N = 5$ 3-algebra is a 3-algebra whose 3-bracket $[\cdot, \cdot, \cdot]$ satisfies the following axioms:*

- (a) $[u, v, w] = [v, u, w]$
- (b) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$
- (c) $[u, v, w] + [v, w, u] + [w, u, v] = 0$.

The following example is inspired by [13]: we just replace $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebras by Lie superalgebras.

Example 5.2 Let \mathfrak{g} be a Lie superalgebra. Define, for $a, b, c \in V := \Pi\mathfrak{g}_{\bar{1}}$, $[a, b, c] = [[a, b], c]$. Then V with this 3-bracket is an $N = 5$ 3-algebra. Indeed, (a) follows from the skew-commutativity of the (super)bracket, and (b) and (c) from the (super) Jacobi identity.

Conversely, any $N = 5$ 3-algebra can be constructed in this way. Namely, let $(N, [\cdot, \cdot, \cdot])$ be an $N = 5$ 3-algebra. Set $\mathfrak{g}_{\bar{1}}(N) = \Pi N$ and let $\mathfrak{g}_0(N)$ be the subalgebra of the Lie superalgebra $\text{End}(N)$ spanned by elements $L_{a,b}$, with $a, b \in N$, defined by:

$$L_{a,b}(c) = [a, b, c].$$

Note that, by property (a) of Definition 5.1, $L_{a,b} = L_{b,a}$. Moreover, $[L_{a,b}, L_{c,d}] = L_{[a,b,c],d} + L_{[a,b,d],c}$. Let $\mathfrak{g}(N) = \mathfrak{g}_0(N) + \mathfrak{g}_{\bar{1}}(N)$ with $[L_{a,b}, c] = L_{a,b}(c) = -[c, L_{a,b}]$, and $[a, b] = L_{a,b}$, for $a, b, c \in \mathfrak{g}_{\bar{1}}(N)$. Then $\mathfrak{g}(N)$ is a Lie superalgebra. Indeed, the skew-commutativity of the bracket follows immediately from the construction. Besides, the (super) Jacobi identity for $\mathfrak{g}(N)$ can be proved as follows: for $a, b, c, d, x \in \mathfrak{g}_{\bar{1}}(N)$,

$$\begin{aligned} [[L_{a,b}, L_{c,d}], x] &= L_{[a,b,c],d}(x) + L_{[a,b,d],c}(x) = [[a, b, c], d, x] + [[a, b, d], c, x] \\ &= [a, b, [c, d, x]] - [c, d, [a, b, x]] = [L_{a,b}, [L_{c,d}, x]] - [L_{c,d}, [L_{a,b}, x]], \end{aligned}$$

where we used property (b) of Definition 5.1; besides,

$$[[L_{a,b}, c], d] = [[a, b, c], d] = L_{[a,b,c],d} = [L_{a,b}, L_{c,d}] - L_{c,[a,b,d]} = [L_{a,b}, [c, d]] - [c, [L_{a,b}, d]].$$

Finally, $[[a, b], c] = [L_{a,b}, c] = [a, b, c] = -[b, c, a] - [a, c, b] = [a, [b, c]] + [b, [a, c]]$, by property (c) of Definition 5.1.

A skew-symmetric bilinear form (\cdot, \cdot) on a finite-dimensional $N = 5$ 3-algebra is called invariant if the 4-linear form $([a, b, c], d)$ on it is invariant under permutations (ab) , (cd) and $(ac)(bd)$ (which generate a dihedral group of order 8).

It is easy to see that if \mathfrak{g} is a finite-dimensional Lie superalgebra, then the restriction of any invariant supersymmetric bilinear form (\cdot, \cdot) on \mathfrak{g} to $\mathfrak{g}_{\bar{1}}$ defines on the $N = 5$ 3-algebra $V = \Pi\mathfrak{g}_{\bar{1}}$ an invariant bilinear form. If, in addition, \mathfrak{g} is a simple Lie superalgebra and the bilinear form is non-degenerate, then \mathfrak{g} is isomorphic to one of Lie superalgebras $psl(m, n)$, $osp(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, or $H(2k)$ [14]. All examples of the corresponding $N = 5$ 3-algebras appear in [2], except for $\mathfrak{g} = H(2k)$. In the latter case the corresponding $N = 5$ 3-algebra is the subspace of odd elements of the Grassmann algebra in $2k$ indeterminates ξ_i with reversed parity, endowed with the following 3-bracket: $[a, b, c] = \{\{a, b\}, c\}$, where $\{a, b\} = \sum_i \frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial \xi_i}$, the invariant bilinear form being $(a, b) = \text{coefficient of } \xi_1 \dots \xi_{2k} \text{ in } ab^*$, where b^* is the Hodge dual of b .

We show, in conclusion, how to associate an $N = 5$ 3-algebra to an $N = 6$ 3-algebra. Let $(L, [\cdot, \cdot, \cdot]_6)$ be an $N = 6$ 3-algebra. Let $T = L + L'$, where $L' = \langle \varphi_x, x \in L \rangle$, $\varphi_x(y, z) = -[y, x, z]_6$. Let $\sigma : T \rightarrow T$ be defined by: $\sigma(z) = -\varphi_z$, $\sigma(\varphi_z) = z$ (cf. Remark 2.4). Then $\sigma^2 = -1$. Now define on T the following 3-bracket ($a, b, c \in T$):

$$[a, b, c]_5 = 0 \text{ if } a, b \in L \text{ or } a, b \in L';$$

$$[a, b, c]_5 = [b, a, c]_5 = [a, \sigma(b), c]_6 = [[a, b], c] \text{ if } a \in L, b \in L'.$$

Then $(T, [\cdot, \cdot, \cdot]_5)$ is an $N = 5$ 3-algebra.

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Classification of linearly compact simple $N=6$ 3-algebras

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Abstract

$N \leq 8$ 3-algebras have recently appeared in N -supersymmetric 3-dimensional Chern-Simons gauge theories. In our previous paper we classified linearly compact simple $N = 8$ n -algebras for any $n \geq 3$. In the present paper we classify linearly compact simple $N = 6$ 3-algebras, using their correspondence with simple linearly compact Lie superalgebras with a consistent short \mathbb{Z} -grading, endowed with a graded conjugation. We also briefly discuss $N = 5$ 3-algebras.

0 Introduction

In recent papers on N -supersymmetric 3-dimensional Chern-Simons gauge theories various types of 3-algebras have naturally appeared (see [12], [3], [4], [1], [2], ...).

Recall that a 3-algebra (also called a triple system [13]) is a vector space V with a ternary (or 3-)bracket $V^{\otimes 3} \rightarrow V$, $a \otimes b \otimes c \mapsto [a, b, c]$.

The 3-algebras that appear in supersymmetric 3-dimensional Chern-Simons theories satisfy certain symmetry conditions and a Jacobi-like identity, usually called the fundamental identity (very much like the Lie algebra bracket).

The simplest among them are 3-Lie algebras, for which the symmetry condition is the total anti-commutativity:

$$(0.1) \quad [a, b, c] = -[b, a, c] = -[a, c, b],$$

and the fundamental identity is:

$$(0.2) \quad [a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]].$$

(Of course, identity (0.2) simply says that for each $a, b \in V$, the endomorphism $D_{a,b}(x) = [a, b, x]$ is a derivation of the 3-algebra V , very much like the Jacobi identity for Lie algebras; in fact, this identity appears already in [13].)

The notion of a 3-Lie algebra generalizes to that of an n -Lie algebra for an arbitrary integer $n \geq 2$ in the obvious way. In this form they were introduced by Filippov in 1985 [10]. It was subsequently proved in Ling's thesis [17] that for each $n \geq 3$ there is only one simple finite-dimensional n -Lie algebra over an algebraically closed field of characteristic 0, by an analysis of the linear Lie algebra spanned by the derivations $D_{a,b}$ (for $n = 3$ this fact was independently proved in [11]). This unique simple n -Lie algebra is the vector product n -algebra O^n in an $n + 1$ -dimensional vector space [10], [17]. Recall that, endowing an $n + 1$ -dimensional vector space V with a non-degenerate symmetric

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bilinear form (\cdot, \cdot) and choosing a basis $\{a_i\}$ and the dual basis $\{a^i\}$, so that $(a_i, a^j) = \delta_{ij}$, the vector product of n vectors from the basis $\{a_i\}$ is defined by:

$$(0.3) \quad [a_{i_1}, \dots, a_{i_n}] = \epsilon_{i_1, \dots, i_{n+1}} a^{i_{n+1}},$$

where $\epsilon_{i_1, \dots, i_{n+1}}$ is a non-zero totally antisymmetric tensor, and extended by n -linearity.

In [9] we extended this classification to linearly compact n -Lie algebras. Recall that a linearly compact n -algebra is a topological n -algebra (i.e., the n -product is continuous) whose underlying topological vector space is linearly compact. The basic examples of linearly compact spaces over a field \mathbb{F} are the spaces of formal power series $\mathbb{F}[[x_1, \dots, x_n]]$, endowed with formal topology, or a direct sum of a finite number of such spaces (these include finite-dimensional vector spaces with discrete topology). Our result is that a complete list of simple linearly compact n -Lie algebras over an algebraically closed field \mathbb{F} of characteristic 0 for $n > 2$ consists of four n -algebras: the $n + 1$ -dimensional O^n and three infinite-dimensional, which we denoted by S^n , W^n , SW^n (see [9] for their construction).

Our method consists in associating to an n -Lie algebra \mathfrak{g} a Lie superalgebra $L = \bigoplus_{j=-1}^{n-1} L_j$ with a consistent \mathbb{Z} -grading, such that $L_{-1} = \Pi\mathfrak{g}$ (i.e., the vector space \mathfrak{g} with odd parity), satisfying the following properties:

$$(0.4) \quad [a, L_{-1}] = 0, a \in L_j, j \geq 0, \quad \text{imply} \quad a = 0 \quad (\text{transitivity}) ;$$

$$(0.5) \quad \dim L_{n-1} = 1 \quad \text{unless all } n\text{-brackets are } 0;$$

$$(0.6) \quad [L_j, L_{-1}] = L_{j-1} \quad \text{for all } j ;$$

$$(0.7) \quad [L_j, L_{n-j-1}] = 0 \quad \text{for all } j .$$

Provided that $n > 2$, the Lie superalgebra $[L, L] = \bigoplus_{j=-1}^{n-2} L_j$ is simple, hence the classification of simple linearly compact n -Lie algebras is thereby reduced to the known classification of simple linearly compact Lie superalgebras [14], [16] and their consistent \mathbb{Z} -gradings [15], [5].

Note that, given a consistently \mathbb{Z} -graded Lie superalgebra $L = \bigoplus_{j=-1}^{n-1} L_j$, satisfying (0.4)–(0.7), we can recover the n -bracket on $\mathfrak{g} = \Pi L_{-1}$ by choosing a non-zero $\mu \in L_{n-1}$ and letting:

$$[a_1, \dots, a_n] = [\dots [\mu, a_1], \dots, a_n] .$$

The 3-Lie algebras appear in $N = 8$ supersymmetric 3-dimensional Chern-Simons theories, hence it is natural to call them the $N = 8$ 3-algebras. The next in the hierarchy of 3-algebras are those which appear in $N = 6$ supersymmetric 3-dimensional Chern-Simons theories (case $N = 7$ reduces to $N = 8$), which we shall call $N = 6$ 3-algebras. They are defined by the following axioms:

$$\begin{aligned} [a, b, c] &= -[c, b, a] ; \\ [a, b, [x, y, z]] &= [[a, b, x], y, z] - [x, [b, a, y], z] + [x, y, [a, b, z]] . \end{aligned}$$

Note that any $N = 8$ 3-algebra is also an $N = 6$ 3-algebra. (It is unclear how to define $N = 6$ n -algebras for $n > 3$.)

The main goal of the present paper is the classification of simple linearly compact $N = 6$ 3-algebras over \mathbb{C} . The method again consists of associating to an $N = 6$ 3-algebra a \mathbb{Z} -graded Lie superalgebra, but in a different way (our construction in [9] uses total anti-commutativity in an essential way).

Given an $N = 6$ 3-algebra \mathfrak{g} , following Palmkvist [P], we associate to \mathfrak{g} a pair $(L = L_{-1} \oplus L_0 \oplus L_1, \varphi)$, where L is a consistently \mathbb{Z} -graded Lie superalgebra with $L_{-1} = \Pi\mathfrak{g}$ and φ is its automorphism, such that the following properties hold:

$$(0.8) \quad \text{transitivity} \quad (0.4);$$

$$(0.9) \quad [L_{-1}, L_1] = L_0;$$

$$(0.10) \quad \varphi(L_j) = L_{-j} \quad \text{and} \quad \varphi^2(a) = (-1)^j a \quad \text{if} \quad a \in L_j.$$

An automorphism φ of the \mathbb{Z} -graded Lie superalgebra L , satisfying (0.10), is called a *graded conjugation*.

It is easy to see that \mathfrak{g} is a simple $N = 6$ 3-algebra if and only if the associated Lie superalgebra L is simple. Thus, this construction reduces the classification of simple linearly compact $N = 6$ 3-algebras to the classification of consistent \mathbb{Z} -gradings of simple linearly compact Lie superalgebras of the form $L = L_{-1} \oplus L_0 \oplus L_1$ (then (0.8) and (0.9) automatically hold), and their graded conjugations φ . The 3-bracket on $\mathfrak{g} = \Pi L_{-1}$ is recovered by letting $[a, b, c] = [[a, \varphi(b)], c]$.

The resulting classification of $N = 6$ 3-algebras is given by Theorem 1.11, and its proof is given in Sections 2, 3, and 4. In Section 5, based on the discussion in [2], we propose a definition and obtain a classification of $N = 5$ 3-algebras, and give a construction of a reduction from $N = 6$ 3-algebras to $N = 5$ 3-algebras.

The base field is an algebraically closed field \mathbb{F} of characteristic zero.

1 Examples of $N = 6$ 3-algebras

Definition 1.1 *An $N = 6$ 3-algebra is a 3-algebra whose 3-bracket $[\cdot, \cdot, \cdot]$ satisfies the following axioms:*

$$(a) \quad [u, v, w] = -[w, v, u]$$

$$(b) \quad [u, v, [x, y, z]] = [[u, v, x], y, z] - [x, [v, u, y], z] + [x, y, [u, v, z]]$$

Example 1.2 Every 3-Lie algebra is an $N = 6$ 3-algebra.

Example 1.3 Let A be an associative algebra and let $*$ be an anti-involution of A , i.e., for every $a \in A$, $(a^*)^* = a$ and for every $a, b \in A$, $(ab)^* = b^*a^*$. Then A with 3-bracket

$$(1.1) \quad [a, b, c] = ab^*c - cb^*a$$

is an $N = 6$ 3-algebra. For instance, if $A = M_n(\mathbb{F})$ and $*$ is the transposition map $^t : a \mapsto a^t$, then the corresponding 3-bracket (1.1) defines on A an $N = 6$ 3-algebra structure. Likewise, if $n = 2k$ and $*$ is the symplectic involution

$$* : a \mapsto J_{2k} a^t J_{2k}^{-1},$$

with $J_{2k} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, then (1.1) defines on $M_n(\mathbb{F})$ an $N = 6$ 3-algebra structure. If $n = 2$, this is in fact a 3-Lie algebra structure.

More generally, consider the vector space $M_{m,n}(\mathbb{F})$ of $m \times n$ matrices with entries in \mathbb{F} , and let $*$: $M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$ be a map satisfying the following property: for $v, u, y \in M_{m,n}(\mathbb{F})$,

$$(1.2) \quad (vu^*y)^* = y^*uv^*.$$

Then $M_{m,n}(\mathbb{F})$ with the 3-bracket (1.1) is an $N = 6$ 3-algebra. (Note that if $m = n$, then property (1.2) implies that $*$ is an anti-involution). As an example we can take $*$ = t ; we denote this 3-algebra by $A^3(m, n; t)$. Likewise, if $m = 2h$ and $n = 2k$, it is easy to check that the map $st : M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$, given by

$$(1.3) \quad st : a \mapsto J_{2k} a^t J_{2h}^{-1},$$

satisfies property (1.2), hence $M_{m,n}(\mathbb{F})$ with the corresponding 3-bracket (1.1) is an $N = 6$ 3-algebra, which we denote by $A^3(m, n; st)$. It is easy to see that the 3-algebra $A^3(2, 2; st)$ is actually an $N=8$ 3-algebra, isomorphic to O^3 .

Lemma 1.4 (a) Let $*$: $M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$ be defined by: $b^* = k^{-1}b^t h$ for some symmetric matrices $h \in GL_m(\mathbb{F})$ and $k \in GL_n(\mathbb{F})$, then the corresponding 3-bracket (1.1) is isomorphic to the one associated to transposition.

(b) Let $^+ : M_{2h,2k}(\mathbb{F}) \rightarrow M_{2k,2h}(\mathbb{F})$ be defined by: $a^+ = H_{2k} a^t H_{2h}^{-1}$ for some skew-symmetric matrices $H_{2i} \in GL_{2i}(\mathbb{F})$, then the corresponding 3-bracket (1.1) is isomorphic to the one associated to map (1.3).

Proof. (a) First of all notice that every symmetric matrix $A \in GL_n(\mathbb{F})$ can be written as the product $B^t B$ where $B \in GL_n(\mathbb{F})$ and B^t is its transpose. Hence let $h = x^t x \in GL_m(\mathbb{F})$ and $k = y^t y \in GL_n(\mathbb{F})$ and consider the map $\varphi : M_{m,n}(\mathbb{F}) \rightarrow M_{n,m}(\mathbb{F})$ defined by $\varphi(u) = x^{-1} u y$. For $a, b, c \in M_{m,n}(\mathbb{F})$, let $[a, b, c] = ab^t c - cb^t a$ and $[a, b, c]^* = ab^* c - cb^* a$. Then we have: $\varphi([a, b, c]) = \varphi(ab^t c - cb^t a) = x^{-1} ab^t c y - x^{-1} cb^t a y = (x^{-1} a y)(y^{-1} b^t x)(x^{-1} c y) - (x^{-1} c y)(y^{-1} b^t x)(x^{-1} a y) = \varphi(a) y^{-1} (y^{-1})^t (\varphi(b))^t x^t x \varphi(c) - \varphi(c) y^{-1} (y^{-1})^t (\varphi(b))^t x^t x \varphi(a) = \varphi(a) k^{-1} (\varphi(b))^t h \varphi(c) - \varphi(c) k^{-1} (\varphi(b))^t h \varphi(a) = [\varphi(a), \varphi(b), \varphi(c)]^*$, and this shows that the 3-brackets $[, ,]$ and $[, ,]^*$ are isomorphic.

(b) Now let $H_{2k} = B^{-1} J_{2k} B$ and $H_{2h} = A^{-1} J_{2h} A$ for some matrices $A \in GL_{2k}$ and $B \in GL_{2h}$ such that $B^{-1} = B^t$, $A^{-1} = A^t$. For $a, b, c \in M_{2h,2k}(\mathbb{F})$, let $[a, b, c]^* = ab^* c - cb^* a$ with $b^* = J_{2k} b^t J_{2h}^{-1}$. Let $\varphi : M_{2h,2k}(\mathbb{F}) \rightarrow M_{2k,2h}(\mathbb{F})$ be defined by $\varphi(x) = A^{-1} x B$. Then we have: $\varphi([a, b, c]^*) = \varphi(ab^* c - cb^* a) = A^{-1} ab^* c B - A^{-1} cb^* a B = (A^{-1} a B)(B^{-1} b^* A)(A^{-1} c B) - (A^{-1} c B)(B^{-1} b^* A)(A^{-1} a B) = \varphi(a) B^{-1} J_{2k} b^t J_{2h}^{-1} A \varphi(c) - \varphi(c) B^{-1} J_{2k} b^t J_{2h}^{-1} A \varphi(a) = \varphi(a) H_{2k} B^t b^t A H_{2h}^{-1} \varphi(c) - \varphi(c) H_{2k} B^t b^t A H_{2h}^{-1} \varphi(a) = \varphi(a) (H_{2k} \varphi(b)^t H_{2h}^{-1}) \varphi(c) - \varphi(c) (H_{2k} \varphi(b)^t H_{2h}^{-1}) \varphi(a) = [\varphi(a), \varphi(b), \varphi(c)]^+$, and this shows that the 3-brackets $[, ,]^*$ and $[, ,]^+$ are isomorphic. \square

Example 1.5 Let us consider the map $\psi : M_{1,2n}(\mathbb{F}) \rightarrow M_{2n,1}(\mathbb{F})$, defined by: $\psi(X Y) = (Y - X)^t$, for $X, Y \in M_{1,2n}$. Then $M_{1,2n}$ with 3-bracket

$$(1.4) \quad [A, B, C] = -AB^t C + CB^t A - C\psi(A)\psi(B)^t$$

is an $N = 6$ 3-algebra, which we denote by $C^3(2n)$.

The $N=6$ 3-algebras $A^3(m, n; t)$ and $C^3(2n)$ were introduced in [4].

Lemma 1.6 Let $[\cdot, \cdot, \cdot]_*$ be the 3-bracket on $M_{1,2n}$ defined by $[a, b, c]_* = -a(kb^t \alpha^{-1})c + c(kb^t \alpha^{-1})a - c\psi(a)\alpha(\psi(b))^t k^{-1}$ for some $\alpha \in \mathbb{F}^\times$ and some symmetric matrix $k \in Sp_{2n}(\mathbb{F})$. Then $[\cdot, \cdot, \cdot]_*$ is isomorphic to (1.4).

Proof. It is convenient to identify $M_{1,2n}(\mathbb{F})$ with the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ \ell & m \end{pmatrix} \in M_{2,2n}$, with $\ell, m \in M_{1,n}$. Under this identification, for $Z \in M_{1,2n}(\mathbb{F})$, $\psi(Z) = J_{2n}Z^tJ_{2n}^{-1}$. Note that we can write $k = yy^t$ for some matrix $y \in Sp_{2n}(\mathbb{F})$ ¹, and, for $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in GL_2(\mathbb{F})$, $h = xx^t$ for some $x \in GL_2(\mathbb{F})$. Consider the map $\varphi : M_{1,2n}(\mathbb{F}) \rightarrow M_{1,2n}(\mathbb{F})$ defined by: $\varphi(u) = xuy^{-1}$ and let $[\cdot, \cdot, \cdot]$ be the 3-bracket (1.4). Then we have: $\varphi([a, b, c]) = -(xay^{-1})yb^tx^{-1}(xcy^{-1}) + (xcy^{-1})yb^tx^{-1}(xay^{-1}) - (xcy^{-1})y\psi(a)\psi(b)^ty^{-1} = -\varphi(a)yb^tx^{-1}\varphi(c) + \varphi(c)yb^tx^{-1}\varphi(a) - \varphi(c)y\psi(a)(\psi(b))^ty^{-1} = -\varphi(a)k\varphi(b)^th^{-1}\varphi(c) + \varphi(c)k\varphi(b)^th^{-1}\varphi(a) - \varphi(c)\psi(\varphi(a))h\psi(\varphi(b))^tk^{-1} = [\varphi(a), \varphi(b), \varphi(c)]_*$, since $y\psi(a)(\psi(b))^ty^{-1} = \psi(\varphi(a))h\psi(\varphi(b))^tk^{-1}$. Indeed we have: $\psi(\varphi(a))h\psi(\varphi(b))^tk^{-1} = J_{2n}(xay^{-1})^tJ_{2n}^{-1}xx^t(J_{2n}(xb y^{-1})^tJ_{2n}^{-1})^t(y^t)^{-1}y^{-1} = J_{2n}(y^{-1})^ta^tx^tJ_{2n}^{-1}xx^tJ_{2n}^{-1}J_{2n}^{-1}(y^t)^{-1}y^{-1} = yJ_{2n}a^tbJ_{2n}^{-1}y^{-1} = y\psi(a)\psi(b)^ty^{-1}$ since $y^tJ_{2n}y = J_{2n}$ and $x^tJ_{2n}^{-1}hJ_{2n}x = I_2$. \square

Example 1.7 Let P be a generalised Poisson algebra with bracket $\{.,.\}$ and derivation D (see [7] for the definition). Let σ be a Lie algebra automorphism of P such that $-\sigma$ is an associative algebra automorphism, $\sigma^2 = 1$ and $\sigma \circ D = -D \circ \sigma$. Then P with the 3-bracket:

$$(1.5) \quad [f, g, h] = \{f, \sigma(g)\}h + \{f, h\}\sigma(g) + f\{\sigma(g), h\} + D(f)\sigma(g)h - f\sigma(g)D(h),$$

is an $N = 6$ 3-algebra.

For example, consider the generalised Poisson algebra $P(m, 0)$ in the (even) indeterminates $p_1, \dots, p_k, q_1, \dots, q_k$ (resp. $p_1, \dots, p_k, q_1, \dots, q_k, t$) if $m = 2k$ (resp. $m = 2k + 1$) endowed with the bracket:

$$(1.6) \quad \{f, g\} = (2 - E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)(g) + \sum_{i=1}^k \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),$$

where $E = \sum_{i=1}^k (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$ (the first two terms in (1.6) vanish if m is even) and the derivation $D = 2 \frac{\partial}{\partial t}$ (which is 0 if m is even). Then the map $\sigma_\varphi : f(p_i, q_i) \mapsto -f(\varphi(p_i), \varphi(q_i))$ (resp. $\sigma_\varphi : f(t, p_i, q_i) \mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i))$), where φ is an involutive linear change of variables (i.e. $\varphi^2 = 1$), multiplying by -1 the 1-form $\sum_i (p_i dq_i - q_i dp_i)$ if m is even (resp. $dt + \sum_i (p_i dq_i - q_i dp_i)$ if m is odd), satisfies the conditions described above, hence the corresponding 3-bracket (1.5) defines on $P(m, 0)$ an $N = 6$ 3-algebra structure. We denote this 3-algebra by $P^3(m; \varphi)$.

Example 1.8 Let $A = \mathbb{F}[[x]]^{\langle 1 \rangle} \oplus \mathbb{F}[[x]]^{\langle 2 \rangle}$ be the direct sum of two copies of the algebra $\mathbb{F}[[x]]$; for $f \in \mathbb{F}[[x]]$, denote by $f^{\langle i \rangle}$ the corresponding element in $\mathbb{F}[[x]]^{\langle i \rangle}$. Set $D = d/dx$ and let $a = (a_{ij}) \in M_{2,2}(\mathbb{F})$. We define the following 3-bracket on A ($i, j = 1$ or 2):

$$[f^{\langle i \rangle}, g^{\langle i \rangle}, h^{\langle i \rangle}] = (-1)^i a_{ij}((fD(h) - D(f)h)g(\varphi(x)))^{\langle i \rangle} \quad \text{for } j \neq i;$$

$$[f^{\langle i \rangle}, g^{\langle j \rangle}, h^{\langle i \rangle}] = (-1)^i a_{jj}((fD(h) - D(f)h)g(\varphi(x)))^{\langle i \rangle} \quad \text{for } j \neq i;$$

$$[f^{\langle 1 \rangle}, g^{\langle j \rangle}, h^{\langle 2 \rangle}] = a_{j1}((fD(g(\varphi(x))) - D(f)g(\varphi(x)))h^{\langle 1 \rangle}) + a_{j2}(f(hD(g(\varphi(x))) - D(h)g(\varphi(x))))^{\langle 2 \rangle},$$

¹As E. Vinberg explained to us, if σ is an anti-involution of a connected reductive group G and S denotes its fixed point set in G , then, by a well-known argument of Cartan, any element $k \in S$ can be represented in the form $k = y\sigma(y)$ for some $y \in G$, provided that S is connected.

and extend it to A by skew-symmetry in the first and third entries. If $a \in SL_2(\mathbb{F})$ and either $a^2 = -1$ and $\varphi = -1$, or $a^2 = 1$ and $\varphi = 1$, then $(A, [\cdot, \cdot, \cdot])$ is an $N = 6$ 3-algebra, which we denote by $SW^3(a)$. Note that if $a^2 = 1$, i.e., up to rescaling, $a = 1$, and $\varphi = 1$, then we get the $N = 8$ 3-algebra SW^3 [9].

Example 1.9 Let $A = \mathbb{F}[[x_1, x_2]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$. We consider the following 3-bracket:

$$(1.7) \quad [f, g, h] = \det \begin{pmatrix} f & \varphi(g) & h \\ D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \end{pmatrix}$$

where φ is an automorphism of the algebra A . If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with 3-bracket (1.7) is an $N = 6$ 3-algebra, which we denote by $S^3(\varphi)$. Note that if $\varphi = 1$, then we get the $N = 8$ 3-algebra S^3 [9]. Clearly, all 3-algebras $S^3(\varphi)$ with $\varphi \neq 1$ are isomorphic to each other.

Example 1.10 Let $A = \mathbb{F}[[x_1, x_2, x_3]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$. Consider the following 3-bracket on A :

$$(1.8) \quad [f, g, h] = \det \begin{pmatrix} D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \\ D_3(f) & D_3(\varphi(g)) & D_3(h) \end{pmatrix}$$

where φ is an automorphism of the algebra A . If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with product (1.8) is an $N = 6$ 3-algebra, which we denote by $W^3(\varphi)$. Note that if $\varphi = 1$, then we get the $N = 8$ 3-algebra W^3 [9]. Clearly, all 3-algebras $W^3(\varphi)$ with $\varphi \neq 1$ are isomorphic to each other.

One can check directly that the above examples are indeed $N = 6$ 3-algebras. However a proof of this without any computations will follow from the connection of $N = 6$ 3-algebras to Lie superalgebras, discussed below. The main result of the paper is the following theorem.

Theorem 1.11 *The following is a complete list of simple linearly compact $N = 6$ 3-algebras over \mathbb{C} :*

- (a) *finite-dimensional:* $A^3(m, n; t)$, $A^3(2m, 2n; st)$, $C^3(2n)$ ($m, n \geq 1$);
- (b) *infinite-dimensional:* $P^3(m; \varphi)$ ($m \geq 1$), $SW^3(a)$, $S^3(\varphi)$, $W^3(\varphi)$.

Proof. Theorem 2.3 from Section 2 reduces the classification in question to that of the pairs (L, σ) , where $L = L_{-1} \oplus L_0 \oplus L_1$ is a simple linearly compact Lie superalgebra with a consistent \mathbb{Z} -grading and σ is a graded conjugation of L . A complete list of possible such $L = L_{-1} \oplus L_0 \oplus L_1$ is given by Remarks 3.2 and 3.4 from Section 3. Finally, a complete list of graded conjugations of these L is given by Propositions 4.3 and 4.8 from Section 4.

By Theorem 2.3(b), the $N = 6$ 3-algebra is identified with ΠL_{-1} , on which the 3-bracket is given by the formula $[a, b, c] = [[a, \sigma(b)], c]$. This formula, applied to the \mathbb{Z} -graded Lie superalgebras L with graded conjugations, described by Proposition 4.3 (a), (b), (c) in the finite-dimensional case produces the 3-algebras $A^3(m, n; t)$, $A^3(2m, 2n; st)$, $C^3(2n)$, respectively, and those, described by Proposition 4.8(a) and (b), (c) and (d), (e) and (f), (g) in the infinite-dimensional case produces the 3-algebras $P^3(m; \varphi)$, $SW^3(a)$, $S^3(\varphi)$, $W^3(\varphi)$, respectively. The fact that all of them are indeed $N = 6$ 3-algebras follows automatically from Theorem 2.3(b). \square

2 Palmkvist's construction

Definition 2.1 Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a Lie superalgebra with a consistent \mathbb{Z} -grading. A graded conjugation of \mathfrak{g} is a Lie superalgebra automorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

1. $\varphi(\mathfrak{g}_j) = \mathfrak{g}_{-j}$
2. $\varphi^2(x) = (-1)^k x$ for $x \in \mathfrak{g}_k$.

Theorem 2.2 Let $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$ be a \mathbb{Z} -graded consistent Lie superalgebra with a graded conjugation φ . Then the 3-bracket

$$[u, v, w] := [[u, \varphi(v)], w]$$

defines on $\Pi \mathfrak{g}_{-1}$ an $N = 6$ 3-algebra structure.

Proof. Since the grading of \mathfrak{g} is consistent, \mathfrak{g}_{-1} and \mathfrak{g}_1 are completely odd and \mathfrak{g}_0 is even. For $u, v, w \in \mathfrak{g}_{-1}$ we thus have: $[u, v, w] := [[u, \varphi(v)], w] = [u, [\varphi(v), w]] = -[[\varphi(v), w], u] = -[[w, \varphi(v)], u] = -[w, v, u]$, which proves property (a) in Definition 1.1.

Besides, for $u, v, x, y, z \in \mathfrak{g}_{-1}$ we have: $[u, v, [x, y, z]] - [x, y, [u, v, z]] = [[u, \varphi(v)], [[x, \varphi(y)], z]] - [[x, \varphi(y)], [[u, \varphi(v)], z]] = [[u, \varphi(v)], [x, \varphi(y)], z] + [[x, \varphi(y)], [u, \varphi(v)], z] - [[x, \varphi(y)], [u, \varphi(v)], z] = [[u, \varphi(v)], [x, \varphi(y)], z] = [[u, \varphi(v)], [x], \varphi(y)], z] + [[[x, [u, \varphi(v)], \varphi(y)], z] = [[u, \varphi(v)], x], \varphi(y)], z] - [[[x, \varphi([u, \varphi(v)], y)], z] = [[u, v, x], y, z] - [x, [v, u], y], z] \quad \square$

We shall now associate to an $N = 6$ 3-algebra T with 3-bracket $[\cdot, \cdot, \cdot]$, a \mathbb{Z} -graded Lie superalgebra $LieT = Lie_{-1}T \oplus Lie_0T \oplus Lie_1T$, as follows. For $x, y \in T$, denote by $L_{x,y}$ the endomorphism of T defined by $L_{x,y}(z) = [x, y, z]$. Besides, for $x \in T$, denote by φ_x the map in $Hom(\Pi T \otimes \Pi T, \Pi T)$ defined by $\varphi_x(y, z) = -[y, x, z]$. Here, as usual, ΠT denotes the vector space T with odd parity.

We let $Lie_{-1}T = \Pi T$, $Lie_0T = \langle L_{x,y} \mid x, y \in T \rangle$, $Lie_1T = \langle \varphi_x \mid x \in T \rangle$, and let $LieT = Lie_{-1}T \oplus Lie_0T \oplus Lie_1T$. Define the map $\sigma : LieT \rightarrow LieT$ by $(x, y, z \in T)$:

$$z \mapsto -\varphi_z, \quad \varphi_z \mapsto z, \quad L_{x,y} \mapsto -L_{y,x}.$$

Theorem 2.3 (a) $LieT$ is a \mathbb{Z} -graded Lie superalgebra with a short consistent grading, satisfying the following two properties:

(i) any non-zero \mathbb{Z} -graded ideal of $LieT$ has a non-zero intersection with both $Lie_{-1}T$ and Lie_1T ;

(ii) $[Lie_{-1}T, Lie_1T] = Lie_0T$.

(b) σ is a graded conjugation of the \mathbb{Z} -graded Lie superalgebra $LieT$ and the 3-product on T is recovered from the bracket on $LieT$ by the formula $[x, y, z] = [[x, \sigma(y)], z]$.

(c) The correspondence $T \rightarrow (LieT, \sigma)$ is bijective and functorial between $N = 6$ 3-algebras and the pairs $(LieT, \sigma)$, where $LieT$ is a \mathbb{Z} -graded Lie superalgebra with a short consistent grading, satisfying properties (i) and (ii), and σ is a graded conjugation of $LieT$.

(d) A 3-algebra T is simple (resp. finite-dimensional or linearly compact) if and only if $LieT$ is.

Proof. For $x, y, z \in T$, we have: $[\varphi_x, z] = -L_{z,x}$, and

$$(2.1) \quad [L_{x,y}, L_{x',y'}] = L_{[x,y,x'],y'} - L_{x',[y,x,y']}.$$

Note that $[L_{x,y}, \varphi_z] = -\varphi_{[y,x,z]}$. Finally, $[[\varphi_x, \varphi_y], z] = [\varphi_x, [\varphi_y, z]] + [\varphi_y, [\varphi_x, z]] = -[\varphi_x, L_{z,y}] - [\varphi_y, L_{z,x}] = [L_{z,y}, \varphi_x] + [L_{z,x}, \varphi_y] = -\varphi_{[y,z,x]} - \varphi_{[x,z,y]} = 0$. Hence $[\varphi_x, \varphi_y] = 0$.

It follows that $LieT$ is indeed a \mathbb{Z} -graded Lie superalgebra, satisfying (ii) and such that any non-zero ideal has a non-zero intersection with $Lie_{-1}T$. It is straightforward to check (b), hence any non-zero ideal of $LieT$ has a non-zero intersection with Lie_1T , which completes the proof of (a).

(c) is clear by construction. Since the simplicity of T , by definition, means that all operators $L_{x,y}$ have no common non-trivial invariant subspace in T , it follows that T is a simple 3-algebra if and only if Lie_0T acts irreducibly on $Lie_{-1}T$. Hence, by the properties (i) and (ii) of $LieT$, T is simple if and only if $LieT$ is simple. The rest of (d) is clear as well. \square

Remark 2.4 $\sigma|_{Lie_0T} = 1$ if and only if T is a 3-Lie algebra.

3 Short gradings and graded conjugations

In this section we shall classify all short gradings of all simple linearly compact Lie superalgebras. We shall describe the short gradings of the classical Lie superalgebras \mathfrak{g} , $\mathfrak{g} \neq Q(n), P(n)$, in terms of linear functions f on the set of roots of \mathfrak{g} . For the description of the root systems of the classical Lie superalgebras, we shall refer to [15]. As for the Lie superalgebra $Q(n)$, we will denote by e_i and f_i the standard Chevalley generators of $Q(n)_{\bar{0}}$, and by \bar{e}_i and \bar{f}_i the corresponding elements in $Q(n)_{\bar{1}}$. Besides, we will identify $P(n)$ with the subalgebra of the Lie superalgebra $SHO(n, n)$ spanned by the following elements: $\{x_i x_j, \xi_i \xi_j : i, j = 1, \dots, n; x_i \xi_j : i \neq j = 1, \dots, n; x_i \xi_i - x_{i+1} \xi_{i+1} : i = 1, \dots, n-1\}$ (cf. [8, §8], and thus describe the \mathbb{Z} -gradings of $P(n)$ as induced by the \mathbb{Z} -gradings of $SHO(n, n)$. Recall that the Lie superalgebra $W(0, n)$ is simple for $n \geq 2$, and for $n = 2$ it is isomorphic to the classical Lie superalgebra $osp(2, 2)$ [14]. Moreover the Lie superalgebra $H(0, n)$ is simple for $n \geq 4$, and for $n = 4$ it is isomorphic to $psl(2, 2)$. Hence, when dealing with $W(0, n)$ (resp. $H(0, n)$) we shall always assume $n \geq 3$ (resp. $n \geq 5$). Likewise, since $W(1, 1) \cong K(1, 2)$ and $S(2, 1) \cong SKO(2, 3; 0)$ [5, §0], when dealing with $W(m, n)$ and $S(m, n)$ we shall always assume $(m, n) \neq (1, 1)$ and $(m, n) \neq (2, 1)$, respectively.

Proposition 3.1 *A complete list of simple finite-dimensional \mathbb{Z} -graded Lie superalgebras with a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same dimension, is, up to isomorphism, as follows:*

1. $A_m, B_m, C_m, D_m, E_6, E_7$ with the \mathbb{Z} -gradings, defined for each s such that $a_s = 1$ by $f(\alpha_s) = 1$, $f(\alpha_i) = 0$ for all $i \neq s$, where $\alpha_1, \dots, \alpha_m$ are simple roots and $\sum_i a_i \alpha_i$ is the highest root;
2. $psl(m, n)$ with the \mathbb{Z} -gradings defined by: $f(\epsilon_1) = \dots = f(\epsilon_k) = 1$, $f(\epsilon_{k+1}) = \dots = f(\epsilon_m) = 0$, $f(\delta_1) = \dots = f(\delta_h) = 1$, $f(\delta_{h+1}) = \dots = f(\delta_n) = 0$, for each $k = 1, \dots, m$ and $h = 1, \dots, n$;
3. $osp(2m+1, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta_j) = 0$ for all j ;
4. $osp(2, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\delta_j) = 0$ for all j ;

5. $osp(2, 2n)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1/2$, $f(\delta_j) = 1/2$ for all j ;
6. $osp(2m, 2n)$, $m \geq 2$, with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta_j) = 0$ for all j ;
7. $osp(2m, 2n)$, $m \geq 2$, with the \mathbb{Z} -grading defined by: $f(\epsilon_i) = 1/2$, $f(\delta_j) = 1/2$ for all i, j ;
8. $D(2, 1; \alpha)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = f(\epsilon_2) = 1/2$, $f(\epsilon_3) = 0$;
9. $F(4)$ with the \mathbb{Z} -grading defined by: $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ for all $i \neq 1$, $f(\delta) = 1$;
10. $Q(n)$ with the gradings defined by: $\deg(e_i) = \deg(\bar{e}_i) = -\deg(f_i) = -\deg(\bar{f}_i) = k_i$, with $k_s = 1$ for some s and $k_i = 0$ for all $i \neq s$;
11. $P(n)$ with $n = 2h \geq 2$ and the gradings of type $(1, \dots, 1, 0, \dots, 0 | 0, \dots, 0, 1, \dots, 1)$ with h 1's and h 0's both in the even and odd part;
12. $H(0, n)$ with the grading of type $(1, 0, \dots, 0, -1)$.

Proof. The claim for simple Lie algebras (the proof of which uses conjugacy of Borel subalgebras) is well known (see e.g. [15]).

In order to classify all short gradings of all classical Lie superalgebras \mathfrak{g} , we shall classify linear functions f on the set of roots of \mathfrak{g} which take values 1, 0 or -1 .

Let $\mathfrak{g} = psl(m, n)$, i.e. $= sl(m, n)$ for $m \neq n$, and $= sl(n, n)/FI_{n,n}$ for $m = n$. Since \mathfrak{g} has (even) roots $\pm(\epsilon_i - \epsilon_j)$ and $\pm(\delta_i - \delta_j)$, a linear function f on the set of roots taking values 0 and ± 1 , is defined, up to a permutation of ϵ_i 's and δ_j 's, by: $f(\epsilon_1) = \dots = f(\epsilon_k) = a$, $f(\epsilon_{k+1}) = \dots = f(\epsilon_m) = a - 1$, $f(\delta_1) = \dots = f(\delta_h) = b$, $f(\delta_{h+1}) = \dots = f(\delta_n) = b - 1$, for some $a, b, k = 0, \dots, m$, $h = 0, \dots, n$. On the other hand, \mathfrak{g} has (odd) roots $\pm(\epsilon_i - \delta_j)$, hence, either $b = a$ or $k = m$ and $b = a + 1$. Since the value of f on the roots is independent of a , we can let $a = 1$.

Now let $\mathfrak{g} = osp(2m + 1, 2n)$. Then \mathfrak{g} has roots δ_i and $2\delta_i$, hence $f(\delta_i) = 0$ for every i . It follows that for every j , either $f(\epsilon_j) = \pm 1$ or $f(\epsilon_j) = 0$. If $f(\epsilon_j) = 0$ for every j , then we get a grading which is not short, hence we can assume, up to equivalence, that $f(\epsilon_1) = 1$. Since \mathfrak{g} has roots $\epsilon_i \pm \epsilon_j$, it follows that $f(\epsilon_j) = 0$ for every $j \neq 1$.

Now let $\mathfrak{g} = osp(2, 2n)$. Then \mathfrak{g} has even roots of the form $\pm 2\delta_i$ and $\pm \delta_i \pm \delta_j$, and odd roots of the form $\pm \epsilon_1 \pm \delta_i$ hence, either $f(\delta_i) = 0$ or $f(\delta_i) = \pm 1/2$. If $f(\delta_i) = 0$ for every i , then $f(\epsilon_1) = 1$. If $f(\delta_k) = 1/2$ for some k , then $f(\delta_i) = 0$ for every i and $f(\epsilon_1) = 1/2$. We hence get two inequivalent short gradings.

Likewise, if $\mathfrak{g} = osp(2m, 2n)$ with $m \geq 2$, either $f(\delta_i) = 0$ for every i , $f(\epsilon_k) = 1$ for some k and $f(\epsilon_j) = 0$ for every $j \neq k$, or $f(\delta_i) = 0$ for every i and $f(\epsilon_j) = 1/2$ for every j .

Now let $\mathfrak{g} = D(2, 1; \alpha)$. Then \mathfrak{g} has roots $\pm 2\epsilon_i$, $i = 1, 2, 3$, hence we may assume that $f(\epsilon_1) = 1/2$. It follows that, up to equivalence, $f(\epsilon_2) = 1/2$ and $f(\epsilon_3) = 0$.

Let $\mathfrak{g} = F(4)$. Then \mathfrak{g} has even roots $\pm \epsilon_i \pm \epsilon_j$, $i \neq j$, $\pm \epsilon_i$, $\pm \delta$, and odd roots $1/2(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)$. It follows that there exists some k such that $f(\epsilon_k) = 1$, hence, $f(\epsilon_i) = 0$ for every $i \neq k$ and $f(\delta) = \pm 1$, i.e., up to equivalence, \mathfrak{g} has only one short grading.

Let $\mathfrak{g} = G(3)$. Since δ and 2δ are roots of \mathfrak{g} , we have $f(\delta) = 0$. Moreover, for every i , either $f(\epsilon_i) = 0$ or $f(\epsilon_i) = \pm 1$. If $f(\epsilon_i) = 0$ for every i , then the grading is not short, hence we may assume, up to equivalence, $f(\epsilon_1) = 1$. It follows that, for $j = 2, 3$, either $f(\epsilon_j) = 1$ or $f(\epsilon_j) = 0$. But this contradicts the linearity of f since $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Hence $G(3)$ has no short gradings.

Let $\mathfrak{g} = Q(n)$. Then any \mathbb{Z} -grading of \mathfrak{g} is defined by setting $\deg(e_i) = \deg(\bar{e}_i) = -\deg(f_i) = -\deg(\bar{f}_i) = k_i \in \mathbb{Z}_{\geq 0}$. Then, since $q(n)_{\bar{0}} \cong A_n$, it is clear that such a grading has depth one if and only if all k_i 's are 0 except for $k_s = 1$ for some s .

Finally, we recall that a complete list of \mathbb{Z} -gradings of depth 1 of all simple linearly compact Lie superalgebras is given in [8, Proposition 8.1]. Then if \mathfrak{g} is a simple finite-dimensional Lie superalgebra which is not classical, or $\mathfrak{g} = P(n)$, we select among these gradings the short ones such that \mathfrak{g}_1 and \mathfrak{g}_{-1} have the same dimension, hence getting gradings 11. and 12. in the statement. \square

Remark 3.2 It follows from Proposition 3.1 that a complete list of simple finite-dimensional \mathbb{Z} -graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same dimension, is, up to isomorphism, as follows:

- $psl(m, n)$ with $m, n \geq 1$, $m + n \geq 2$, with the grading $f(\epsilon_1) = \dots = f(\epsilon_m) = 1$, $f(\delta_1) = \dots = f(\delta_n) = 0$;
- $osp(2, 2n)$, $n \geq 1$, with the grading $f(\delta_i) = 0$ for all i , $f(\epsilon_1) = 1$.

Notice that $P(n)$ has, up to equivalence, a unique consistent \mathbb{Z} -grading, i.e., the grading of type $(1, \dots, 1|0, \dots, 0)$. In this grading one has: $\mathfrak{g}_{\bar{0}} \cong sl_{n+1}$, $\mathfrak{g}_{-1} \cong \Lambda^2(\mathbb{F}^{n+1})^*$ and $\mathfrak{g}_1 \cong S^2\mathbb{F}^{n+1}$, where \mathbb{F}^{n+1} denotes the standard sl_{n+1} -module, hence \mathfrak{g}_1 and \mathfrak{g}_{-1} have different dimension. Finally, no short grading of the Lie superalgebra $Q(n)$ is consistent since in this case $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$ -module.

Proposition 3.3 *A complete list of simple linearly compact infinite-dimensional \mathbb{Z} -graded Lie superalgebras with a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth and the same size, is, up to isomorphism, as follows:*

- $S(1, 2)$ with the grading of type $(0|1, 0)$;
- $S(1, 2)$ with the grading of type $(0|1, 1)$;
- $H(2k, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- $K(2k + 1, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$;
- $E(1, 6)$ with the grading of type $(0|1, 0, 0, -1, 0, 0)$.

Proof. All \mathbb{Z} -gradings of depth 1 of all infinite-dimensional linearly compact simple Lie superalgebras are listed, up to isomorphism, in [8, Proposition 9.1]. Among these gradings we first select those which are short, hence getting the following list:

- 1) $W(m, n)$, with $m > 0$, $n \geq 1$, $(m, n) \neq (1, 1)$, with the grading of type $(0, \dots, 0|1, 0, \dots, 0)$;
- 2) $W(m, n)$, with $m > 0$, $n \geq 1$, $(m, n) \neq (1, 1)$, with the grading of type $(0, \dots, 0|-1, 0, \dots, 0)$;

- 3) $S(m, n)$, with $m > 1$ and $n \geq 1$, $(m, n) \neq (2, 1)$, or $m = 1$ and $n \geq 2$, with the grading of type $(0, \dots, 0|1, 0, \dots, 0)$;
- 4) $S(1, 2)$ with the grading of type $(0|1, 1)$;
- 5) $H(2k, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- 6) $K(2k + 1, n)$ with the grading of type $(0, \dots, 0|1, 0, \dots, 0, -1)$;
- 7) $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- 8) $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$;
- 9) $E(1, 6)$ with the grading of type $(0|1, 0, 0, -1, 0, 0)$.

Let us consider $W(m, n)$ with $n \geq 1$ and the grading of type $(0, \dots, 0|1, 0, \dots, 0)$. Then $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$ and $\mathfrak{g}_1 = \langle \xi_1 \frac{\partial}{\partial x_i}, \xi_1 \frac{\partial}{\partial \xi_j} | i = 1, \dots, m, j = 2, \dots, n \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$. Therefore \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth equal to m but \mathfrak{g}_{-1} has size 2^{n-1} and \mathfrak{g}_1 has size $(m + n - 1)2^{n-1}$. It follows that for $m > 0$, $n \geq 1$ and $(m, n) \neq (1, 1)$, \mathfrak{g}_{-1} and \mathfrak{g}_1 do not have the same size. Likewise case 2) is ruled out.

Now let us consider $S(m, n)$ with $n \geq 1$ and the grading of type $(0, \dots, 0|1, 0, \dots, 0)$. Then $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n)$ and $\mathfrak{g}_1 = \{f \in \langle \xi_1 \frac{\partial}{\partial x_i}, \xi_1 \frac{\partial}{\partial \xi_j} | i = 1, \dots, m, j = 2, \dots, n \rangle \otimes \mathbb{F}[[x_1, \dots, x_m]] \otimes \Lambda(\xi_2, \dots, \xi_n) \mid \text{div}(f) = 0\}$. Therefore \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth equal to m but \mathfrak{g}_{-1} has size 2^{n-1} and \mathfrak{g}_1 has size $(m + n - 2)2^{n-1}$. It follows that for $m > 1$, $(m, n) \neq (2, 1)$, or $m = 1$ and $n \geq 2$, \mathfrak{g}_{-1} and \mathfrak{g}_1 do not have the same size unless $m = 1$ and $n = 2$. \square

Remark 3.4 It follows from Proposition 3.3 that a complete list of simple infinite-dimensional \mathbb{Z} -graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that \mathfrak{g}_{-1} and \mathfrak{g}_1 have the same growth and size, is, up to isomorphism, as follows:

- $S(1, 2)$ with the grading of type $(0|1, 1)$;
- $H(2k, 2)$ with the grading of type $(0, \dots, 0|1, -1)$;
- $K(2k + 1, 2)$ with the grading of type $(0, \dots, 0|1, -1)$;
- $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$;
- $SKO(2, 3; \beta)$ with the grading of type $(0, 0|1, 1, 1)$.

4 Classification of graded conjugations

In this section we shall classify all graded conjugations σ of all \mathbb{Z} -graded simple linearly compact Lie superalgebras \mathfrak{g} with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In Lemma 4.7 and Proposition 4.8 we shall assume that $\mathbb{F} = \mathbb{C}$. This assumption can be removed with a little extra work.

Remark 4.1 If φ is an automorphism of \mathfrak{g} preserving the grading and σ is a graded conjugation, then $\varphi\sigma\varphi^{-1}$ is again a graded conjugation which is equivalent to σ . Indeed, we have: $\varphi\sigma\varphi^{-1}\varphi\sigma\varphi^{-1} = \varphi\mu\varphi^{-1} = \mu$, where $\mu|_{\mathfrak{g}_k} = (-1)^k \text{id}$.

Remark 4.2 If φ is an involution of \mathfrak{g} preserving the grading and commuting with σ , then $\varphi\sigma$ is again a graded conjugation.

Proposition 4.3 *The following is a complete list, up to equivalence, of graded conjugations of all simple finite-dimensional Lie superalgebras:*

$$(a) \ \mathfrak{g} = sl(m, n)/\mathbb{F}I\delta_{m,n}: \sigma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^t & c^t \\ -b^t & -d^t \end{pmatrix}.$$

$$(b) \ \mathfrak{g} = sl(2h, 2k)/\mathbb{F}I\delta_{2h,2k}: \sigma_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^{st} & c^{st} \\ -b^{st} & -d^{st} \end{pmatrix}, \text{ where } st \text{ denotes the symplectic transposition defined by (1.3).}$$

$$(c) \ \mathfrak{g} = osp(2, 2n): \sigma_1.$$

Proof. The Lie superalgebra $sl(m, n)$ has a short consistent grading such that $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, where $tr\alpha = tr\delta$, \mathfrak{g}_{-1} is the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$, and \mathfrak{g}_1 is the set of matrices of the form $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$.

For $m \neq n$ every automorphism of $sl(m, n)$ is either of the form $Ad \ diag(A, B)$ for some matrices $A \in GL_m(\mathbb{F})$, $B \in GL_n(\mathbb{F})$, or of the form $Ad \ diag(A, B) \circ \sigma_1$ [19]. Note that $\sigma_1^2 = Ad \ diag(I_m, -I_n)$ and $\sigma_1^3 = \sigma_1^{-1} = Ad \ diag(I_m, -I_n) \circ \sigma_1$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(m, n)$, we have: $Ad \ diag(A, B) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AaA^{-1} & AbB^{-1} \\ BcA^{-1} & BdB^{-1} \end{pmatrix}$, hence every automorphism $Ad \ diag(A, B)$ maps \mathfrak{g}_1 (resp. \mathfrak{g}_{-1}) to itself and does not define a graded conjugation of \mathfrak{g} . Let $\varphi_{A,B} = Ad \ diag(A, B) \circ \sigma_1$. Then $\varphi_{A,B} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -Aa^tA^{-1} & Ac^tB^{-1} \\ -Bb^tA^{-1} & -Bd^tB^{-1} \end{pmatrix}$. If $\varphi_{A,B}$ is a graded conjugation of \mathfrak{g} , then, by definition, $\varphi_{A,B}|_{\mathfrak{g}_0}^2 = 1$, hence $A^tA^{-1} = \lambda I_m$ and $B^tB^{-1} = \rho I_n$, for some $\lambda, \rho \in \mathbb{F}$. Besides, since $\varphi_{A,B}|_{\mathfrak{g}_{\bar{1}}}^2 = -1$, we have $\rho = \lambda$. It follows that $A^t = \lambda A$ hence, by transposing both sides of the equality, $A = \lambda A^t = \lambda^2 A$, i.e., $\lambda^2 = 1$. Therefore, either $A^t = A$ and $B^t = B$, or $A^t = -A$, $B^t = -B$ and m and n are even (since A and B are invertible). The thesis then follows from Lemma 1.4.

If $m = n$, in addition to the automorphisms described above, $sl(n, n)/\mathbb{F}I_{n,n}$ has automorphisms of the form $Ad \ diag(A, B) \circ \Pi$, $Ad \ diag(A, B) \circ \Pi \circ \sigma_1$ and $Ad \ diag(A, B) \circ \sigma_1 \circ \Pi$, where $A, B \in GL_n(\mathbb{F})$, and Π is defined as follows: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(n, n)$, $\Pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ [19]. Note that $\sigma_1 \circ \Pi \circ \sigma_1 = \Pi$ and $\Pi \circ \sigma_1 \circ \Pi = \sigma_1^{-1}$. The automorphisms of the form $Ad \ diag(A, B) \circ \Pi \circ \sigma_1$ and $Ad \ diag(A, B) \circ \sigma_1 \circ \Pi$ map \mathfrak{g}_1 (resp. \mathfrak{g}_{-1}) to itself, hence they do not define graded conjugations of \mathfrak{g} . Let $\psi_{A,B} = Ad \ diag(A, B) \circ \Pi$. Then $\psi_{A,B} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AdA^{-1} & AcB^{-1} \\ BbA^{-1} & BaB^{-1} \end{pmatrix}$. It follows that $\psi_{A,B}|_{\mathfrak{g}_0}^2 = 1$ if and only if $AB = BA = \lambda I_n$. As a consequence, $\psi_{A,B}|_{\mathfrak{g}_{\bar{1}}}^2 = 1$, hence $\psi_{A,B}$ does not define a graded conjugation.

The Lie superalgebra $osp(2, 2n)$ has a short consistent grading such that $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where $A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in \mathbb{F}$, and D lies in the Lie algebra $sp(2n)$,

defined by J_{2n} , \mathfrak{g}_{-1} is the set of matrices of the form $\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & a & b \\ \hline b^t & 0 & & \\ -a^t & 0 & & \end{array} \right)$, and \mathfrak{g}_1 is the set of

matrices of the form $\left(\begin{array}{cc|cc} & & a & b \\ & & 0 & 0 \\ \hline 0 & b^t & & \\ 0 & -a^t & & \end{array} \right)$, with $a, b \in M_{1,n}$.

Every automorphism of $osp(2, 2n)$ is either of the form $Ad \, diag(A, B)$ for some matrices $A = diag(\alpha, \alpha^{-1})$, $\alpha \in \mathbb{F}^\times$, $B \in Sp_{2n}(\mathbb{F})$, or of the form $Ad \, diag(A, B) \circ \sigma_1$ [19]. One can easily check that every automorphism of the form $Ad \, diag(A, B)$ sends \mathfrak{g}_{-1} (resp. \mathfrak{g}_1) to itself, hence it does not define a graded conjugation of \mathfrak{g} . Let $\Phi_{A,B} = Ad \, diag(A, B) \circ \sigma_1$. Then, using the same arguments as for the automorphisms $\varphi_{A,B}$ of the Lie superalgebra $sl(m, n)$, one can show that $\Phi_{A,B}$ defines a graded conjugation of $osp(2, 2n)$ if and only if B is a symmetric matrix. Then the result follows from Lemma 1.6. \square

Remark 4.4 If \mathfrak{g} is a simple infinite-dimensional linearly compact Lie superalgebra, then $Aut \, \mathfrak{g}$ contains a maximal reductive subgroup which is explicitly described in [6, Theorem 4.2]. We shall denote this subgroup by G . We point out that any reductive subgroup of $Aut \, \mathfrak{g}$ is conjugate into G , in particular any finite order element of $Aut \, \mathfrak{g}$ is conjugate to an element of G .

Example 4.5 The grading of type $(0, \dots, 0|1, -1)$ of $\mathfrak{g} = H(2k, 2)$ (resp. $K(2k+1, 2)$) is short. Let $A = \mathbb{F}[[p_1, \dots, p_k, q_1, \dots, q_k]]$ (resp. $A = \mathbb{F}[[t, p_1, \dots, p_k, q_1, \dots, q_k]]$). We have:

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle \xi_2 \rangle \otimes A, \\ \mathfrak{g}_0 &= (\langle 1, \xi_1 \xi_2 \rangle \otimes A) / \mathbb{F}1 \text{ (resp. } \langle 1, \xi_1 \xi_2 \rangle \otimes A), \\ \mathfrak{g}_1 &= \langle \xi_1 \rangle \otimes A. \end{aligned}$$

For every linear involutive change of variables φ , multiplying by -1 the 1-form $\sum_{i=1}^k (p_i dq_i - q_i dp_i)$ (resp. $dt + \sum_{i=1}^k (p_i dq_i - q_i dp_i)$), the following map is a graded conjugation of \mathfrak{g} :

$$(4.1) \quad \begin{aligned} f(p_i, q_i) &\mapsto -f(\varphi(p_i), \varphi(q_i)) & (\text{resp. } f(t, p_i, q_i) &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i))) \\ f(p_i, q_i) \xi_1 \xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i)) \xi_1 \xi_2 & (\text{resp. } f(t, p_i, q_i) \xi_1 \xi_2 &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_1 \xi_2) \\ f(p_i, q_i) \xi_1 &\mapsto f(\varphi(p_i), \varphi(q_i)) \xi_2 & (\text{resp. } f(t, p_i, q_i) \xi_1 &\mapsto f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_2) \\ f(p_i, q_i) \xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i)) \xi_1 & (\text{resp. } f(t, p_i, q_i) \xi_2 &\mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i)) \xi_1). \end{aligned}$$

Example 4.6 Let $\mathfrak{g} = S(1, 2)$, $SHO(3, 3)$, or $SKO(2, 3; 1)$. Then the algebra of outer derivations of \mathfrak{g} contains $sl_2 = \langle e, h, f \rangle$, with $e = \xi_1 \xi_2 \frac{\partial}{\partial x}$ and $h = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$ if $\mathfrak{g} = S(1, 2)$, $e = \xi_1 \xi_3 \frac{\partial}{\partial x_2} - \xi_2 \xi_3 \frac{\partial}{\partial x_1} - \xi_1 \xi_2 \frac{\partial}{\partial x_3}$ and $h = \sum_{i=1}^3 \xi_i \frac{\partial}{\partial \xi_i}$ if $\mathfrak{g} = SHO(3, 3)$, $e = \xi_1 \xi_2 \tau$ and $h = 1/2(\tau - x_1 \xi_1 - x_2 \xi_2)$ if $\mathfrak{g} = SKO(2, 3; 1)$. Let us denote by G_{out} the subgroup of $Aut \, \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(f))$ and $\exp(ad(h))$. We recall that $G_{out} \subset G$, where G is the subgroup of $Aut \, \mathfrak{g}$ introduced in Remark 4.4 [6, Remark 2.2, Theorem 4.2]. We shall denote by U_- the one parameter group of automorphisms $\exp(ad(tf))$, and by G_{inn} the subgroup of G consisting of inner automorphisms. Finally, H will denote the subgroup of $Aut \, \mathfrak{g}$ consisting of invertible changes of variables multiplying the volume form (resp. the even supersymplectic form) by a constant if $\mathfrak{g} = S(1, 2)$ (resp. $\mathfrak{g} = SHO(3, 3)$), or the odd supercontact form by a function if $\mathfrak{g} = SKO(2, 3; 1)$ (see [6, Theorem 4.5]).

The gradings of type $(0|1, 1)$, $(0, 0, 0|1, 1, 1)$ and $(0, 0|1, 1, 1)$ of $\mathfrak{g} = S(1, 2)$, $SHO(3, 3)$ and $SKO(2, 3; 1)$, respectively, are short, and the subspaces \mathfrak{g}_i 's are as follows:

$\mathfrak{g} = S(1, 2)$:

$$\begin{aligned}\mathfrak{g}_{-1} &= \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes \mathbb{F}[[x]] \\ \mathfrak{g}_0 &= \{f \in \langle \frac{\partial}{\partial x}, \xi_i \frac{\partial}{\partial \xi_j} | i, j = 1, 2 \rangle \otimes \mathbb{F}[[x]], \text{div}(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \xi_i \frac{\partial}{\partial x}, \xi_1 \xi_2 \frac{\partial}{\partial \xi_i} | i = 1, 2 \rangle \otimes \mathbb{F}[[x]], \text{div}(f) = 0\}.\end{aligned}$$

$\mathfrak{g} = SHO(3, 3)$:

$$\begin{aligned}\mathfrak{g}_{-1} &= \mathbb{F}[[x_1, x_2, x_3]]/\mathbb{F}1 \\ \mathfrak{g}_0 &= \{f \in \langle \xi_1, \xi_2, \xi_3 \rangle \otimes \mathbb{F}[[x_1, x_2, x_3]] | \Delta(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \xi_i \xi_j, i, j = 1, 2, 3 \rangle \otimes \mathbb{F}[[x_1, x_2, x_3]] | \Delta(f) = 0\}.\end{aligned}$$

$\mathfrak{g} = SKO(2, 3; 1)$:

$$\begin{aligned}\mathfrak{g}_{-1} &= \mathbb{F}[[x_1, x_2]] \\ \mathfrak{g}_0 &= \{f \in \langle \xi_1, \xi_2, \tau \rangle \otimes \mathbb{F}[[x_1, x_2]] | \text{div}_1(f) = 0\} \\ \mathfrak{g}_1 &= \{f \in \langle \tau \xi_i, \xi_1 \xi_2 | i = 1, 2 \rangle \otimes \mathbb{F}[[x_1, x_2]] | \text{div}_1(f) = 0\}.\end{aligned}$$

In all these cases the map $s = \exp(\text{ad}(e)) \exp(\text{ad}(-f)) \exp(\text{ad}(e))$ is a graded conjugation of \mathfrak{g} : for $z \in \mathfrak{g}_{-1}$, $s(z) = [e, z]$; for $z \in \mathfrak{g}_1$, $s(z) = -[f, z]$, for $z \in \mathfrak{g}_0$, $s(z) = z$. Note that each of the above gradings can be extended to $\text{Der } \mathfrak{g} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{a} \supset \mathfrak{sl}_2$, so that e has degree 2, h has degree 0, and f has degree -2 .

Lemma 4.7 *If \mathfrak{g} is one of the following \mathbb{Z} -graded Lie superalgebras:*

1. $S(1, 2)$ with the grading of type $(0|1, 1)$,
2. $SKO(2, 3; 1)$ with the grading of type $(0, 0|1, 1, 1)$,
3. $SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$,

and σ is a graded conjugation of \mathfrak{g} , then σ is conjugate to an automorphism of the form $s \circ \exp(\text{ad}(th)) \circ \varphi$, for some $t \in \mathbb{F}$ and some $\varphi \in G_{\text{inn}}$ such that $\varphi^2 = 1$.

Proof. Let us first assume $\mathfrak{g} = S(1, 2)$ with the grading of type $(0|1, 1)$, or $\mathfrak{g} = SKO(2, 3; 1)$ with the grading of type $(0, 0|1, 1, 1)$. By [6, Remark 4.6], if ψ is an automorphism of \mathfrak{g} lying in G , then either $\psi \in U_-H \cap G$ or $\psi \in U_-sH \cap G$. Note that $U_-H \cap G = U_-(H \cap G)$ and $U_-sH \cap G = U_-s(H \cap G)$, since $U_- \subset G$ and $s \in G$ [6, Theorem 4.2]. Here $H \cap G$ is the subgroup of $\text{Aut } \mathfrak{g}$ generated by $\exp(\text{ad}(e))$, $\exp(\text{ad}(h))$ and G_{inn} . Note that $G_{\text{inn}} \subset \exp(\text{ad}(\mathfrak{g}_0))$. Let $\psi \in U_-(H \cap G)$. Then $\psi = \exp(\text{ad}(tf))\psi_0$ for some $t \in \mathbb{F}$ and some $\psi_0 \in H \cap G$. For $x \in \mathfrak{g}_1$, we have: $\psi(x) = \exp(\text{ad}(tf))(\psi_0(x)) = \psi_0(x) + t[f, \psi_0(x)]$, since $\psi_0(x) \in \mathfrak{g}_1$. In particular, $\psi(x) \notin \mathfrak{g}_{-1}$. Now let σ be a graded conjugation of \mathfrak{g} . Then we may assume, up to conjugation, that σ lies in G . Since σ exchanges \mathfrak{g}_1 and \mathfrak{g}_{-1} , by the observation above $\sigma \in U_-s(H \cap G)$, i.e., $\sigma = \exp(\text{ad}(tf))s\varphi_0\varphi_1$ for some $t \in \mathbb{F}$, some $\varphi_1 \in G_{\text{inn}}$ and some φ_0 lying in the subgroup generated by $\exp(\text{ad}(h))$ and $\exp(\text{ad}(e))$. We can assume $\varphi_0 = \exp(\text{ad}(\beta e)) \exp(\text{ad}(\alpha h))$ for some $\alpha, \beta \in \mathbb{F}$, i.e., $\sigma = \exp(\text{ad}(tf))s \exp(\text{ad}(\beta e)) \exp(\text{ad}(\alpha h))\varphi_1$. Since $\varphi_1(\mathfrak{g}_1) = \mathfrak{g}_1$ and $\varphi_1(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$, for $x \in \mathfrak{g}_1$ we have:

$$\sigma(x) = -\exp(\alpha)[f, \varphi_1(x)] = s \circ \exp(\text{ad}(\alpha h)) \circ \varphi_1(x).$$

For $x \in \mathfrak{g}_{-1}$ we have:

$$(4.2) \quad \sigma(x) = \exp(-\alpha)([e, \varphi_1(x)] + (t - \beta)\varphi_1(x)).$$

Notice that if $x \in \mathfrak{g}_{-1}$, then $[e, \varphi_1(x)] \in \mathfrak{g}_1$ and $\varphi_1(x) \in \mathfrak{g}_{-1}$. Since $\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_1$, we have

$$(4.2) = \exp(-\alpha)[e, \varphi_1(x)] = s \circ \exp(ad(\alpha h)) \circ \varphi_1(x).$$

Therefore $\sigma = s \circ \exp(ad(\alpha h)) \circ \varphi_1$. Now notice that $s \circ \exp(ad(\alpha h)) = \exp(ad(-\alpha h)) \circ s$, $\exp(ad(\alpha h)) \circ \varphi_1 = \varphi_1 \circ \exp(ad(\alpha h))$, and $s \circ \varphi_1 = \varphi_1 \circ s$. It follows that $\sigma^2 = s \circ \exp(ad(\alpha h)) \circ \varphi_1 \circ s \circ \exp(ad(\alpha h)) \circ \varphi_1 = s \circ \exp(ad(\alpha h)) \exp(ad(-\alpha h)) \circ s \circ \varphi_1^2 = s^2 \circ \varphi_1^2$, therefore $\varphi_1^2 = 1$.

Now let $\mathfrak{g} = SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$. As in the previous cases, if σ is a graded conjugation of \mathfrak{g} , then $\sigma \in U_-s(H \cap G)$. Here $H \cap G$ is the subgroup of $Aut \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(h))$, $\exp(ad(\Phi))$, and G_{inn} , where $\Phi = \sum_{i=1}^3 (-x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ and G_{inn} is generated by $\exp(ad(x_i \xi_j))$ with $i, j = 1, 2, 3$, $i \neq j$, and is thus isomorphic to SL_3 . Note that Φ commutes with G_{inn} and h . We may hence assume $\sigma = \exp(ad(tf))s\varphi_0\varphi_1$, for some $t \in \mathbb{F}$, some $\varphi_1 \in G_{inn}$ and some φ_0 lying in the subgroup generated by $\exp(ad(e))$, $\exp(ad(h))$, $\exp(ad(\Phi))$, i.e., $\varphi_0 = \exp(ad(\gamma e)) \circ \exp(ad(\beta\Phi)) \circ \exp(ad(\alpha h))$, for some $\alpha, \beta, \gamma \in \mathbb{F}$. Arguing as for $S(1, 2)$ and $SKO(2, 3; 1)$, one shows that, in fact, $\sigma = s \circ \exp(ad(\beta\Phi)) \circ \exp(ad(\alpha h)) \circ \varphi_1$. Besides, the following commutation relations hold: $s \circ \exp(ad(\alpha h)) = \exp(ad(-\alpha h)) \circ s$, $\exp(ad(\alpha h)) \circ \varphi_1 = \varphi_1 \circ \exp(ad(\alpha h))$, $s \circ \varphi_1 = \varphi_1 \circ s$, $\exp(ad(\beta\Phi)) \circ s|_{\mathfrak{g}_{-1}} = \exp(3\beta)s \circ \exp(ad(\beta\Phi))|_{\mathfrak{g}_{-1}}$, $\exp(ad(\beta\Phi)) \circ s|_{\mathfrak{g}_1} = \exp(-3\beta)s \circ \exp(ad(\beta\Phi))|_{\mathfrak{g}_1}$. It follows that, since $\sigma^2 = s^2$, we have: $\exp(3\beta) \exp(ad(\beta\Phi))^2 \varphi_1^2|_{\mathfrak{g}_{-1}} = 1$, $\exp(-3\beta) \exp(ad(\beta\Phi))^2 \varphi_1^2|_{\mathfrak{g}_1} = 1$. Note that G_{inn} acts on \mathfrak{g}_{-1} by the standard action of vector fields on functions. In particular $V = \langle x_1, x_2, x_3 \rangle$ is stabilized by this action and $\exp(ad(\Phi))$ acts on V by scalar multiplication by $\exp(-1)$. It follows that if $F \in SL_3$ is the matrix of the action of φ_1 on V , then $\exp(\beta)I_3 F^2 = I_3$, hence $\exp(3\beta) = \exp(3\beta) \det(F)^2 = 1$. It follows that $(\exp(ad(\beta\Phi)) \circ \varphi_1)^2 = 1$, moreover, we can assume $\beta = 0$, since $\exp(\beta)I_3 \in SL_3$. Hence $\sigma = s \circ \exp(ad(\alpha h)) \circ \varphi_1$, for some $\alpha \in \mathbb{F}$ and some $\varphi_1 \in G_{inn}$ such that $\varphi_1^2 = 1$. \square

Proposition 4.8 *The following is a complete list, up to equivalence, of graded conjugations of all simple infinite-dimensional linearly compact Lie superalgebras \mathfrak{g} :*

- a) $\mathfrak{g} = H(2k, 2)$: σ is the automorphism of \mathfrak{g} defined by (4.1).
- b) $\mathfrak{g} = K(2k + 1, 2)$: σ is the automorphism of \mathfrak{g} defined by (4.1).
- c) $\mathfrak{g} = S(1, 2)$: s .
- d) $\mathfrak{g} = S(1, 2)$: $\sigma = s \circ \exp(ad(\alpha h')) \circ \varphi_0$, where $\exp(2\alpha) = -1$, $h' = 2x \frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$, φ_0 lies in the SL_2 -subgroup of G_{inn} , and $\varphi_0^2 = -1$.
- e) $\mathfrak{g} = SKO(2, 3; 1)$: s .
- f) $\mathfrak{g} = SKO(2, 3; 1)$: $\sigma = s \circ \exp(ad(\alpha h'))$, where $\exp(2\alpha) = -1$ and $h' = x_1 \xi_1 + x_2 \xi_2 + \tau$.
- g) $\mathfrak{g} = SHO(3, 3)$: $\sigma = s \circ \varphi$ with $\varphi \in G_{inn}$ such that $\varphi^2 = 1$.

Proof. By definition of graded conjugation, \mathfrak{g} is, up to isomorphism, one of the \mathbb{Z} -graded Lie superalgebras listed in Remark 3.4. Let $\mathfrak{g} = H(2k, 2)$ with the grading of type $(0, \dots, 0|1, -1)$ (see Example 4.5), and let σ be a graded conjugation of \mathfrak{g} . By Remark 4.4 we can assume that $\sigma \in G = \mathbb{F}^\times (Sp_{2k} \times O_2)$ [6, Theorem 4.2]. Note that G consists of linear changes of variables preserving the symplectic form up to multiplication by a non-zero scalar. Since σ exchanges \mathfrak{g}_{-1}

and \mathfrak{g}_1 , and $\sigma^2|_{\mathfrak{g}_{-1}} = -1$, we have: $\sigma(\xi_1) = a\xi_2$ and $\sigma(\xi_2) = -\frac{1}{a}\xi_1$, for some $a \in \mathbb{F}^\times$, hence, up to multiplication by a scalar, we may assume that $a = 1$. It follows that, for $f \in \mathbb{F}[[p_i, q_i]]$, $\sigma(f) = \sigma([\xi_1, f\xi_2]) = [\xi_2, \sigma(f\xi_2)] = [\xi_2, \tilde{f}\xi_1] = \tilde{f}$ for some $\tilde{f} \in \mathbb{F}[[p_i, q_i]]$, i.e., $\sigma(f\xi_2) = \sigma(f)\xi_1$. Likewise, $\sigma(f\xi_1) = -\sigma(f)\xi_2$, and $\sigma(f\xi_1\xi_2) = \sigma(f)\xi_1\xi_2$. Besides, for $f, g \in \mathbb{F}[[p_i, q_i]]$ $\sigma([f\xi_1, g\xi_2]) = \sigma(fg + [f, g]\xi_1\xi_2) = -[\sigma(f)\xi_2, \sigma(g)\xi_1] = -\sigma(f)\sigma(g) + [\sigma(f), \sigma(g)]\xi_1\xi_2$. Hence $\sigma(fg) = -\sigma(f)\sigma(g)$, i.e., $-\sigma$ is an automorphism of $\mathbb{F}[[p_i, q_i]]$ as an associative algebra. It follows that σ is defined as follows:

$$(4.3) \quad \begin{aligned} f(p_i, q_i) &\mapsto -f(\varphi(p_i), \varphi(q_i)) \\ f(p_i, q_i)\xi_1\xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i))\xi_1\xi_2 \\ f(p_i, q_i)\xi_1 &\mapsto f(\varphi(p_i), \varphi(q_i))\xi_2 \\ f(p_i, q_i)\xi_2 &\mapsto -f(\varphi(p_i), \varphi(q_i))\xi_1 \end{aligned}$$

for some linear change of even variables φ . Since $\sigma(\xi_1) = \xi_2$ and $\sigma(\xi_2) = -\xi_1$, σ multiplies the odd part $d\xi_1 d\xi_2$ of the symplectic form by -1 , hence φ multiplies the even part of the symplectic form by -1 . Moreover, $\sigma^2(f(p_i, q_i)) = f(\varphi^2(p_i), \varphi^2(q_i))$, hence $\varphi^2 = 1$. This concludes the proof of a). The same arguments prove b). In this case, one has $\sigma \circ \frac{\partial}{\partial t} = -\frac{\partial}{\partial t} \circ \sigma$.

Let $\mathfrak{g} = SKO(2, 3; \beta)$ with $\beta \neq 1$ and the grading of type $(0, 0|1, 1, 1)$. By [6, Theorem 4.2], G is generated by $\exp(ad(\tau + x_1\xi_1 + x_2\xi_2))$ and G_{inn} . Note that G_{inn} is contained in $\exp(ad(\mathfrak{g}_0))$, hence no automorphism of \mathfrak{g} exchanges \mathfrak{g}_1 and \mathfrak{g}_{-1} . It follows that \mathfrak{g} has no graded conjugations.

Let $\mathfrak{g} = S(1, 2)$ and let σ be a graded conjugation of \mathfrak{g} . Then, by Lemma 4.7, $\sigma = s \circ \exp(ad(th)) \circ \varphi$ for some $t \in \mathbb{F}$ and some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$. The group G_{inn} is generated by $\exp(ad(h'))$, $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_2}))$, $\exp(ad(\xi_2 \frac{\partial}{\partial \xi_1}))$ and $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}))$, hence we may write $\varphi = \exp(ad(\alpha h'))\varphi_0$ for some $\alpha \in \mathbb{F}$ and some φ_0 in the SL_2 -subgroup of G_{inn} generated by $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_2}))$, $\exp(ad(\xi_2 \frac{\partial}{\partial \xi_1}))$ and $\exp(ad(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}))$. Note that $\varphi(\frac{\partial}{\partial x}) = \exp(-2\alpha)\frac{\partial}{\partial x}$, therefore $\exp(2\alpha) = \pm 1$ since $\varphi^2 = 1$. Besides, if $z \in \mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes \mathbb{F}[[x]]$, then $\varphi^2(z) = \exp(-2\alpha)\varphi_0^2(z)$, therefore either

$$i) \exp(2\alpha) = 1 \text{ and } \varphi_0^2|_{\mathfrak{g}_{-1}} = 1;$$

or

$$ii) \exp(2\alpha) = -1 \text{ and } \varphi_0^2|_{\mathfrak{g}_{-1}} = -1.$$

In case $i)$ we have $\varphi_0|_{\mathfrak{g}_{-1}} = \pm 1$, since $\varphi_0 \in SL_2$. Then $\varphi_0|_{\mathfrak{g}_0} = 1$ since φ_0 acts on \mathfrak{g}_0 via the adjoint action. It follows that $\sigma|_{\mathfrak{g}_0} = 1$, since $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h'))\varphi_0$, $\exp(ad(\alpha h'))|_{\mathfrak{g}_0} = 1$ since $\exp(2\alpha) = 1$, $\exp(ad(th))|_{\mathfrak{g}_0} = 1$ and $s|_{\mathfrak{g}_0} = 1$. By Remark 2.4 and the classification of 3-Lie algebras obtained in [9], we conclude that σ is conjugate to s .

In case $ii)$ φ_0 corresponds to a 2×2 matrix of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ such that $a^2 + bc = -1$.

The corresponding element $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h'))\varphi_0$ acts on \mathfrak{g}_{-1} as follows:

$$\begin{aligned} x^r \frac{\partial}{\partial \xi_1} &\mapsto \exp(2r\alpha) \exp(-t - \alpha) (-arx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_1} - brx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_2} + ax^r\xi_2 \frac{\partial}{\partial x} - bx^r\xi_1 \frac{\partial}{\partial x}); \\ x^r \frac{\partial}{\partial \xi_2} &\mapsto \exp(2r\alpha) \exp(-t - \alpha) (-crx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_1} + arx^{r-1}\xi_1\xi_2 \frac{\partial}{\partial \xi_2} + cx^r\xi_2 \frac{\partial}{\partial x} + ax^r\xi_1 \frac{\partial}{\partial x}). \end{aligned}$$

It follows that, up to rescaling, we may assume that $t = 0$ hence getting d).

In order to classify the graded conjugations of $\mathfrak{g} = SKO(2, 3; 1)$ we argue in a similar way as for $S(1, 2)$. Namely, let σ be such a map, then, by Lemma 4.7, $\sigma = s \circ \exp(ad(th)) \circ \varphi$ for some $t \in \mathbb{F}$

and some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$. The group G_{inn} is generated by $\exp(ad(h'))$, $\exp(ad(x_1\xi_2))$, $\exp(ad(x_2\xi_1))$ and $\exp(ad(x_1\xi_1 - x_2\xi_2))$, hence we may write $\varphi = \exp(ad(\alpha h'))\varphi_0$ for some $\alpha \in \mathbb{F}$ and some φ_0 in the SL_2 -subgroup of G_{inn} generated by $\exp(ad(x_1\xi_2))$, $\exp(ad(x_2\xi_1))$ and $\exp(ad(x_1\xi_1 - x_2\xi_2))$. Note that this SL_2 -subgroup acts on $\mathfrak{g}_{-1} = \mathbb{F}[[x_1, x_2]]$ via the standard action of vector fields on functions, and stabilizes the subspaces of $\mathbb{F}[[x_1, x_2]]$ consisting of homogeneous polynomials of fixed degree. We have: $\varphi(1) = \exp(ad(\alpha h'))(1) = \exp(-2\alpha)$, therefore, since $\varphi^2 = 1$, either $\exp(2\alpha) = 1$ or $\exp(2\alpha) = -1$. If $\exp(2\alpha) = 1$, then, for $f \in \mathfrak{g}_{-1}$, $\varphi(f) = \varphi_0(f)$; if $\exp(2\alpha) = -1$, then, for $f \in \mathfrak{g}_{-1}$, $\varphi(f) = -\varphi_0(f^-)$, where $f^-(x_1, x_2) = f(-x_1, -x_2)$. It follows that $\varphi_0^2|_{\mathfrak{g}_{-1}} = 1$. In particular, if $V = \langle x_1, x_2 \rangle$, then $\varphi_0|_V = \pm 1$, i.e., $\varphi_0 = \exp(ad(A(x_1\xi_1 - x_2\xi_2)))$ with $A \in \mathbb{F}$ such that $\exp(A) = \pm 1$. It follows that $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h')) \exp(ad(A(x_1\xi_1 - x_2\xi_2)))$, for some $t, \alpha, A \in \mathbb{F}$ such that $\exp(2\alpha) = \pm 1$, $\exp(A) = \pm 1$. We now consider the restriction of σ to \mathfrak{g}_0 . We have: $\sigma|_{\mathfrak{g}_0} = \exp(ad(\alpha h')) \exp(ad(A(x_1\xi_1 - x_2\xi_2)))|_{\mathfrak{g}_0}$, since $s|_{\mathfrak{g}_0} = 1$ and $\exp(ad(th))|_{\mathfrak{g}_0} = 1$. It is then easy to check that either

$$i) \exp(2\alpha) \exp(A) = 1 \text{ and } \sigma|_{\mathfrak{g}_0} = 1;$$

or

$$ii) \exp(2\alpha) \exp(A) = -1.$$

In case $i)$, by Remark 2.4 and the classification of 3-Lie algebras obtained in [9], σ is conjugate to s . In case $ii)$, $\sigma = s \circ \exp(ad(th)) \circ \exp(ad(\alpha h')) \exp(ad(A(x_1\xi_1 - x_2\xi_2)))$ acts on $\mathfrak{g}_{-1} = \mathbb{F}[[x_1, x_2]]$ as follows:

$$\begin{aligned} f &\mapsto -\exp(-t)s(f^-), & \text{if } \exp(A) = 1, \exp(-2\alpha) = -1; \\ f &\mapsto \exp(-t)s(f^-), & \text{if } \exp(A) = -1, \exp(-2\alpha) = 1. \end{aligned}$$

Therefore, changing the sign if necessary, we may assume that we are in the first case and in this case we may assume, up to rescaling, that $A = 0 = t$, hence getting f).

Let $\mathfrak{g} = SHO(3, 3)$ with the grading of type $(0, 0, 0|1, 1, 1)$. Then, by Lemma 4.7, $\sigma = s \circ \exp(ad(th)) \circ \varphi$ for some $\varphi \in G_{inn}$ such that $\varphi^2 = 1$, and some $t \in \mathbb{F}$. For $a \in \mathfrak{g}_{-1}$, we have:

$$\sigma(a) = s(\exp(ad(th))(\varphi(a))) = \exp(-t)s(\varphi(a)) = \exp(-t)[e, \varphi(a)],$$

since $\varphi(a) \in \mathfrak{g}_{-1}$. Up to rescaling, we can thus assume that $t = 0$, hence getting g). Note that in this case $G_{inn} \cong SL_3$. \square

Remark 4.9 It is proved in [9] that there are no simple linearly compact $N = 8$ 3-superalgebras, which are not 3-algebras. On the contrary, there are many simple linearly compact $N = 6$ 3-superalgebras beyond 3-algebras. We are planning to classify them in a subsequent publication.

5 $N = 5$ 3-algebras

Based on the discussion in [2], the following seems to be a right definition of an $N = 5$ 3-algebra.

Definition 5.1 *An $N = 5$ 3-algebra is a 3-algebra whose 3-bracket $[\cdot, \cdot, \cdot]$ satisfies the following axioms:*

- (a) $[u, v, w] = [v, u, w]$
- (b) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$

$$(c) \quad [u, v, w] + [v, w, u] + [w, u, v] = 0.$$

The following example is inspired by [13]: we just replace $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebras by Lie superalgebras.

Example 5.2 Let \mathfrak{g} be a Lie superalgebra. Define, for $a, b, c \in V := \Pi \mathfrak{g}_{\bar{1}}$, $[a, b, c] = [[a, b], c]$. Then V with this 3-bracket is an $N = 5$ 3-algebra. Indeed, (a) follows from the skew-commutativity of the (super)bracket, and (b) and (c) from the (super) Jacobi identity.

Conversely, any $N = 5$ 3-algebra can be constructed in this way. Namely, let $(N, [\cdot, \cdot, \cdot])$ be an $N = 5$ 3-algebra. Set $\mathfrak{g}_{\bar{1}}(N) = \Pi N$ and let $\mathfrak{g}_{\bar{0}}(N)$ be the subalgebra of the Lie superalgebra $End(N)$ spanned by elements $L_{a,b}$, with $a, b \in N$, defined by:

$$L_{a,b}(c) = [a, b, c].$$

Note that, by property (a) of Definition 5.1, $L_{a,b} = L_{b,a}$. Moreover, $[L_{a,b}, L_{c,d}] = L_{[a,b,c],d} + L_{[a,b,d],c}$. Let $\mathfrak{g}(N) = \mathfrak{g}_{\bar{0}}(N) + \mathfrak{g}_{\bar{1}}(N)$ with $[L_{a,b}, c] = L_{a,b}(c) = -[c, L_{a,b}]$, and $[a, b] = L_{a,b}$, for $a, b, c \in \mathfrak{g}_{\bar{1}}(N)$. Then $\mathfrak{g}(N)$ is a Lie superalgebra. Indeed, the skew-commutativity of the bracket follows immediately from the construction. Besides, the (super) Jacobi identity for $\mathfrak{g}(N)$ can be proved as follows: for $a, b, c, d, x \in \mathfrak{g}_{\bar{1}}(N)$,

$$\begin{aligned} [[L_{a,b}, L_{c,d}], x] &= L_{[a,b,c],d}(x) + L_{[a,b,d],c}(x) = [[a, b, c], d, x] + [[a, b, d], c, x] \\ &= [a, b, [c, d, x]] - [c, d, [a, b, x]] = [L_{a,b}, [L_{c,d}, x]] - [L_{c,d}, [L_{a,b}, x]], \end{aligned}$$

where we used property (b) of Definition 5.1; besides,

$$[[L_{a,b}, c], d] = [[a, b, c], d] = L_{[a,b,c],d} = [L_{a,b}, L_{c,d}] - L_{c,[a,b,d]} = [L_{a,b}, [c, d]] - [c, [L_{a,b}, d]].$$

Finally, $[[a, b], c] = [L_{a,b}, c] = [a, b, c] = -[b, c, a] - [a, c, b] = [a, [b, c]] + [b, [a, c]]$, by property (c) of Definition 5.1.

A skew-symmetric bilinear form (\cdot, \cdot) on a finite-dimensional $N = 5$ 3-algebra is called invariant if the 4-linear form $([a, b, c], d)$ on it is invariant under permutations (ab) , (cd) and $(ac)(bd)$ (which generate a dihedral group of order 8).

It is easy to see that if \mathfrak{g} is a finite-dimensional Lie superalgebra, then the restriction of any invariant supersymmetric bilinear form (\cdot, \cdot) on \mathfrak{g} to $\mathfrak{g}_{\bar{1}}$ defines on the $N = 5$ 3-algebra $V = \Pi \mathfrak{g}_{\bar{1}}$ an invariant bilinear form. If, in addition, \mathfrak{g} is a simple Lie superalgebra and the bilinear form is non-degenerate, then \mathfrak{g} is isomorphic to one of Lie superalgebras $psl(m, n)$, $osp(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, or $H(2k)$ [14]. All examples of the corresponding $N = 5$ 3-algebras appear in [2], except for $\mathfrak{g} = H(2k)$. In the latter case the corresponding $N = 5$ 3-algebra is the subspace of odd elements of the Grassmann algebra in $2k$ indeterminates ξ_i with reversed parity, endowed with the following 3-bracket: $[a, b, c] = \{\{a, b\}, c\}$, where $\{a, b\} = \sum_i \frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial \xi_i}$, the invariant bilinear form being $(a, b) = \text{coefficient of } \xi_1 \dots \xi_{2k} \text{ in } ab^*$, where b^* is the Hodge dual of b .

We show, in conclusion, how to associate an $N = 5$ 3-algebra to an $N = 6$ 3-algebra. Let $(L, [\cdot, \cdot, \cdot]_6)$ be an $N = 6$ 3-algebra. Let $T = L + L'$, where $L' = \langle \varphi_x, x \in L \rangle$, $\varphi_x(y, z) = -[y, x, z]_6$. Let $\sigma : T \rightarrow T$ be defined by: $\sigma(z) = -\varphi_z$, $\sigma(\varphi_z) = z$ (cf. Remark 2.4). Then $\sigma^2 = -1$. Now define on T the following 3-bracket ($a, b, c \in T$):

$$[a, b, c]_5 = 0 \text{ if } a, b \in L \text{ or } a, b \in L';$$

$$[a, b, c]_5 = [b, a, c]_5 = [a, \sigma(b), c]_6 = [[a, b], c] \text{ if } a \in L, b \in L'.$$

Then $(T, [\cdot, \cdot, \cdot]_5)$ is an $N = 5$ 3-algebra.

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